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Zusammenfassung / Abstract:

The aim of this paper is to give a short proof of the Saito-Kurokawa lift for Hermitian modular forms along the lines we gave in an earlier paper. The proof uses a converse theorem as was initially proven by Imai yet avoiding the framework of spectral analysis.

The proof heavily relies on the observation, that the partial Mellin transform of the Saito-Kurokawa lift of the cusp forms from Kojimas plus space coincides with a certain Siegel theta lift of f matched with a real analytic Eisenstein series. The functional equation of the Eisenstein series then implies the desired functional equation for the partial Mellin transform which in turn proves the lift to be a Hermitian modular form.

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A note on the Saito-Kurokawa lift for Hermitian forms

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1 Introduction

There is a well-known correspondence between classical modular forms contained in the Kohnen space and the Maass Spezialschar of Siegel modular forms, called the Saito-Kurokawa-Lift. Kojima [15] and Krieg [13] found an Hermitian analogue of this lifting.

In [24] Sugano generalized this lifting to holomorphic cusp forms on $\mathrm{SO}(2, m+2)$ also using Jacobi forms. Remark that $m = 1$ refers to Siegel- and $m = 2$ to Hermitian modular forms.

A different proof for the Saito-Kurokawa lift for Siegel modular forms was given by Duke and Imamoglu in [7] using a converse theorem of Imai [12]. In [20] we applied the method of Duke and Imamoglu to the Hermitian case and also reproved the corresponding lift by a converse theorem.

Recently we gave a simpler proof for the Saito-Kurokawa lift, see [17] also using the converse theorem of Imai, but avoiding the analysis of the spectral Koecher-Maass series.

The purpose of the present paper is to generalize this proof to $\mathrm{SO}(2, 4)$ which we refer to as the Hermitian case.

2 Hermitian Saito-Kurokawa Lift

2.1 Hermitian modular forms

The Hermitian upper-half space is defined as

$$\mathbf{H} = \{Z = X + iY, X, Y \in M(2, \mathbb{R}) : (Z - \overline{Z}^t)/i \text{ positive definite}\}.$$

The Hermitian symplectic group $U(2, 2) = \left\{ M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in M_4(\mathbb{C}); \overline{M}^t J M = J \right\}$, $J = \begin{pmatrix} O_2 & -I_2 \\ I_2 & O_2 \end{pmatrix}$, acts on the Hermitian upper half-space by $\begin{pmatrix} A & B \\ C & D \end{pmatrix} Z = (AZ + B)(CZ + D)^{-1}$.

Let $K = \mathbf{Q}(i)$ be the Gaussian number field, $\mathcal{O} = \mathbf{Z}[i]$ the ring of integers of K , $\mathcal{D}^{-1} = (2i)^{-1}\mathcal{O}$ the inverse different and $\chi_K = \begin{pmatrix} -4 \\ \end{pmatrix}$.

We denote by $\Gamma_2 = U(2, 2) \cap M_4(\mathcal{O})$ the full Hermitian modular group of degree two. An Hermitian modular form of weight k and degree 2 is a holomorphic function $F(Z)$ on \mathbf{H} such that

$$F(MZ) = \det^k(CZ + D)F(Z),$$

for any $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ in Γ_2 .

Let

$$L = \left\{ T = \begin{pmatrix} a & \alpha \\ \bar{\alpha} & d \end{pmatrix} : a, d \in \mathbb{Z}, \alpha \in \mathcal{D}^{-1} \right\},$$

which is the dual lattice of the lattice of Hermitian matrices with respect to the trace form. We also call the elements of L half integral hermitian matrices. It is well known, that any Hermitian modular form can be expanded in a Fourier series

$$F(Z) = \sum_{0 \leq T \in L} A(T) e(\text{tr } TZ)$$

where as usual $e(z) = e^{2\pi iz}$. Recall that a matrix is called half integral, if a, d, α are integers.

We use the notation $T \geq 0$ and $T > 0$ for positive semidefinite or definite hermitian matrices, respectively.

The Fourier coefficients satisfy the unimodularity property

$$A(T, F) = A([U]T, F)$$

for any $U \in \text{SL}(2, \mathcal{O})$ where U acts via $[U]T = UT\bar{U}^t$. To see this, just remark, that $e(\text{tr } T[U]Z) = e(\text{tr } [\bar{U}^t]TZ)$ and therefore $F(Z) = F([\bar{U}^{-1}]Z)$ implies

$$\left[\sum_{0 \leq T \in L} A(T) e(\text{tr } TZ) \right] = \sum_{0 \leq T \in L} A(T) e(\text{tr } [U^{-1}]TZ) = \sum_{0 \leq T \in L} A([U]T) e(\text{tr } TZ).$$

F is called a cusp form if $A(T) = 0$ provided $\det T = 0$.

2.2 Saito-Kurokawa lift and Kojima's plus space

Let k be a natural number divisible by 4. Take a cusp form $f(z)$ of weight $k - 1$, character χ_K for $\Gamma_0(4)$ belonging to the plus space S_{k-1}^* in the sense of Kojima [15], i.e.

$$f(z) = \sum_{l \geq 1, \chi_K(l) \neq 1} c(l) e(lz) \in S_{k-1}(\Gamma_0(4), \chi_K),$$

where $z \in \mathbb{H}^2$ which shall denote the usual upper halfspace.

Put $\alpha^*(l) = (-2i)c(l)/(\chi_K(-l) + 1)$,

$$e(T) := \max\{q \in \mathbf{N}; q^{-1}T \in L_2^+\}$$

and define a function $F(Z)$ on \mathbf{H} by

$$F(Z) = \sum_{0 < T \in L} \left(\sum_{d|e(T)} d^{k-1} \alpha^*((4 \det T)/d^2) \right) e(\text{tr}(TZ)). \quad (1)$$

$F(Z)$ is called the Hermitian Saito-Kurokawa lift and we have the following

Theorem 2.1. *$F(Z)$ is a Hermitian modular form of weight k on the full Hermitian modular group Γ_2 .*

As mentioned above, the aim of this paper is to give a further (and shorter) proof of Theorem 2.1, using Imai's converse theorem in a more direct fashion.

3 Imai's converse theorem

3.1 Unimodular invariant Fourier series

Take a Fourier series $F(Z) = \sum_{0 < T \in L} A(T, F) e(\text{tr}(TZ))$ on \mathbf{H} such that

$$A(T, F) = A([U]T, F) \quad (2)$$

for any $U \in \text{SL}(2, \mathcal{O})$. Such a Fourier series we call unimodular invariant. We suppose in addition, that

$$A(T, F) = \mathcal{O}((\det T)^a)$$

with a positive constant a . Under these assumptions we have shown in [20] analogous results as in [12], namely that the series $F(Z)$ converges absolutely and uniformly in any domain $Y \geq Y_0 > \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ and $F(Z)$ is bounded on this region. Here $Y = \frac{1}{2i}(Z - Z^*)$.

Further we have shown that for such a unimodular invariant Fourier series the estimate

$$|F(iY)| \leq (C_1(\det Y)^{-l-1} + C_2(\det Y)^{-l}) e^{-C_3\sqrt{\det Y}}$$

with positive constants C_i, l is valid.

From (2) follows, that for any $U \in \text{SL}(2, \mathcal{O})$

$$F(i[U]Y) = F(iY).$$

There is the following well-known bijective correspondence between positive definite hermitian matrices $W > 0$ with $\det W = 1$ and the three-dimensional hyperbolic space, which can be realized as a subset of quaternions

$$\mathbb{H}^3 = \{u + jv : u \in \mathbb{C}, v > 0\}.$$

given by a Cholesky decomposition of W :

Lemma 3.1. *Let W be a positive definite hermitian matrices with $\det W = 1$ then*

$$W = \omega \bar{\omega}^t \quad (3)$$

with uniquely determined $\omega = \begin{pmatrix} \sqrt{r} & z/\sqrt{r} \\ 0 & 1/\sqrt{r} \end{pmatrix}$ where $w = z + jr \in \mathbb{H}^3$.

Furthermore the correspondence $\phi : W \rightarrow w$ is compatible with the action of $\text{SL}(2, \mathbb{C})$ in the sense that

$$\phi([U]W) = Uw$$

for $U \in \text{SL}(2, \mathbb{C})$.

Especially if $V = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ then $\phi(\bar{W}^{-1}) = \phi([V]W) = -1/w$. We extend ϕ to \mathcal{P}_2 by $\phi(T) := \phi(T/\sqrt{\det T})$.

3.2 Partial Mellin transform and converse theorem

We define as in [12] the partial Mellin transform of F and as a consequence from the above estimates one obtains

Proposition 3.1. *If F is defined by a unimodular Fourier series which satisfies $A(T) = 0$, if $\det T = 0$ and $A(T) \ll (\det T)^a$, $a > 0$, then for $W \in \mathcal{SP}$ the partial Mellin transform*

$$\tilde{F}(W, s) = \int_0^\infty F(iu^{\frac{1}{2}}W)u^{s-1}du \quad (4)$$

exists and it is holomorphic in s for $\Re s$ sufficiently large. Also we can apply Mellin inversion

$$F(it^{\frac{1}{2}}W) = \frac{1}{2\pi i} \int_{\Re s=c} \tilde{F}(W, s)t^{-s}ds$$

for some $c \in \mathbb{R}$.

The unimodularity assumption yields $\tilde{F}([\gamma]W, s) = \tilde{F}(W, s)$ for all $\gamma \in \mathrm{SL}(2, \mathcal{O})$. Therefore we can consider $\tilde{F}(W, s)$ as a modular form on \mathbb{H}^3 under the identification $w = \phi(W)$. Due to this correspondence the partial Mellin transform can be regarded as a function of w and by abuse of notation we may also write $\tilde{F}(w, s)$ instead of $\tilde{F}(W, s)$.

In order that a function F , which is defined by a Fourier series as above, is also a Hermitian modular form of weight k it suffices that it satisfies

$$F(iY^{-1}) = (-1)^k \det^k(Y)F(iY) \quad (5)$$

for $iY \in \mathbf{H}$. This follows as in the work of A. Krieg [14], Lemma 1.6, p.48 and Lemma 1.7, p.79.

We shall use this in the proof of the following converse theorem for the Hermitian case. Essentially the statement is a modified version of Imai's result in [12].

Theorem 3.1 (Imai). *Let F be defined by a unimodular convergent Fourier series as in (2) where the Fourier coefficients are of at most polynomial growth and $A(T) = 0$, if $\det T = 0$. If for each $w \in \mathbb{H}^3$ the partial Mellin transform $\tilde{F}(w, s)$ fulfills the following three conditions:*

- i) it is entire as a function of s*
- ii) it tends to zero as $\Im s \rightarrow \pm\infty$ uniformly in every vertical strip*
- iii) it satisfies the functional equation*

$$\tilde{F}(w, s) = (-1)^k \tilde{F}(w, k - s) \quad (6)$$

then $F(Z)$ is a Hermitian cusp form of degree 2 and weight k .

Proof. Let $t > 0$. We use the Mellin inversion formula, which is valid for $\Re s > c$, for some suitable $c \in \mathbb{R}$ and obtain

$$\begin{aligned}
F(it^{\frac{1}{2}}W) &= \frac{1}{2\pi i} \int_{\Re s=c} \tilde{F}(W, s)t^{-s} ds \\
&= \frac{(-1)^k}{2\pi i} \int_{\Re s=c} \tilde{F}(W, k-s)t^{-s} ds \\
&= \frac{(-1)^k}{2\pi i} \int_{\Re s=k-c} \tilde{F}(w, s)t^{-k+s} ds \\
&= \frac{(-1)^k t^{-k}}{2\pi i} \int_{\Re s=k-c} \tilde{F}(W, s) \left(\frac{1}{t}\right)^{-s} ds \\
&= (-1)^k t^{-k} F(it^{-\frac{1}{2}}W).
\end{aligned}$$

For the last equality we move the path of integration back to $\Re s = c$, which is possible because $\tilde{F}(W, s)$ is entire and tends to zero as $\Im s \rightarrow \pm\infty$ uniformly in every vertical strip. One should notice that

$$W^{-1} = \left[\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \right] \overline{W}.$$

Since

$$\tilde{F}\left(\left[\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \right] \overline{W}, s\right) = \tilde{F}(\overline{W}, s)$$

a similar computation as above gives

$$F(it^{-\frac{1}{2}}W^{-1}) = (-1)^k t^k F(it^{\frac{1}{2}}\overline{W}). \quad (7)$$

Since we assume that F is given by a Fourier series, we have

$$F(it^{\frac{1}{2}}\overline{W}) = \sum_{0 < T \in L} A(T, F) e(it^{\frac{1}{2}} \text{tr}(T\overline{W})).$$

Notice that for our choice of T and W we have $\text{tr}(T\overline{W}) = \text{tr}(\overline{TW})$ and further L is closed with respect to complex conjugation, therefore

$$F(it^{\frac{1}{2}}\overline{W}) = F(it^{\frac{1}{2}}W).$$

We can conclude from (7) that for $Y > 0, \det Y > 0$

$$F(iY^{-1}) = (-1)^k (\det Y)^k F(iY).$$

The statement of the theorem now follows. (Cf. the remark after (5)).

□

In the present paper we shall give a proof for Theorem 2.1 by showing the functional equation for the partial Mellin transform without referring to spectral analysis. The proof heavily relies on the observation, that the partial Mellin transform of the Saito-Kurokawa lift of the cusp form $f \in S_{k-1}^*$ coincides with a certain theta lift of f matched with an Eisenstein series. The functional equation of the Eisenstein series then implies the desired functional equation for the partial Mellin transform.

4 The Siegel theta series and Kojima's plus space

4.1 Siegel's theta series

We consider the lattice $\Lambda = \mathbb{Z}^4$ equipped with the symmetric bilinear form

$$2\Omega = \begin{pmatrix} 0 & 0 & 0 & 4 \\ 0 & -2 & 0 & 0 \\ 0 & 0 & -2 & 0 \\ 4 & 0 & 0 & 0 \end{pmatrix}.$$

Then $\Omega(\lambda) = \lambda^t \Omega \lambda$ is the corresponding quadratic form of signature $(1, 3)$. Let D be such that

$$\Omega = D^t \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} D.$$

For our purposes we choose

$$D = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & -1 \end{pmatrix}.$$

Then $D^t D$ is called a majorant for Ω . Of course there is no unique choice for D . It is at once seen, that if $S \in \text{SO}(1, 3)$ then D can be replaced by SD and all majorants in the above sense are of the form $D^t S^t S D$ for some fixed special majorant D . The group $\text{SO}(\Omega)$ acts transitively on the space of majorants in an obvious way, providing it thus with the structure of a symmetric space.

We also consider the dual lattice

$$\hat{\Lambda} = \{\mu \in \mathbb{Q}^4 : 2\Omega\mu \in \Lambda\}$$

and the discriminant group $\mathbf{D} = \hat{\Lambda}/\Lambda$.

As in [17] we define the vector valued Siegel theta series related to Λ . For this introduce a unitary vector space of dimension $|\mathbf{D}|$ with a unitary basis $v_{\mathbf{r}}$ indexed by $\mathbf{r} \in \mathbf{D}$. Subject to this basis we build a vector-valued theta series by

$$\Theta(z, \Omega^+) = \sum_{\mathbf{r} \in \mathbf{D}} \Theta(\mathbf{r}, z, \Omega^+) v_{\mathbf{r}}$$

where for $\mathbf{r} \in \mathbf{D}$, $z = x + iy \in \mathbb{H}$ and any majorant Ω^+

$$\Theta(\mathbf{r}, z, \Omega^+) := y \sum_{\mu \equiv \mathbf{r} \pmod{\Lambda}} e(x\Omega(\mu) + iy\Omega^+(\mu)).$$

This theta series comes along with a transformation formula which can be derived using the Weil representation as in Shintani [?]. (See also [?].) Siegel stated this formula by direct computations in [23], Hilfssatz 1,

$$\Theta(\gamma z, w) = j_{-1}(\gamma, z) \chi(\gamma) \Theta(z, w)$$

with $j_{-1}(\gamma, z) = \left(\frac{cz+d}{c\bar{z}+d}\right)^{-1/2}$ and a unitary matrix $\chi(\gamma) = (\chi_{\mathbf{r},\mathbf{s}}(\gamma))_{\mathbf{r},\mathbf{s} \in \mathbf{D}}$.

For $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}(2, \mathbb{Z})$ one finds

$$\chi_{\mathbf{r},\mathbf{s}}(\gamma) = \begin{cases} e(\frac{1}{4})|\det(2c\Omega)|^{-\frac{1}{2}} \sum_{g \in \Lambda/c\Lambda} e\left(\frac{a\Omega[g+\mathbf{r}] - 2s^t\Omega(g+\mathbf{r}) + d\Omega[\mathbf{s}]}{c}\right), & c \neq 0 \\ e(ab\Omega[\mathbf{r}]) & c = 0, \mathbf{ar} = \mathbf{s} \\ 0 & c = 0, \mathbf{ar} \neq \mathbf{s} \end{cases}$$

For future purposes we state here the values of χ for the generators $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ and $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ of the modular group. Write $\mathbf{r} = (r_a, r_0, r_1, r_c)^t$ and $\mathbf{s} = (s_a, s_0, s_1, s_c)^t$. With this we obtain

i)

$$\chi_{\mathbf{r},\mathbf{s}}\left(\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\right) = \frac{1}{8}e\left(\frac{1}{4}\right)e(-4s_c r_a - 4s_a r_c + 2s_0 r_0 + 2s_1 r_1) \quad (8)$$

ii)

$$\chi_{\mathbf{r},\mathbf{r}}\left(\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\right) = e(4r_a r_c - r_0^2 - r_1^2) \quad (9)$$

iii)

$$\chi_{\mathbf{r},\mathbf{s}}\left(\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\right) = 0, \quad \mathbf{r} \neq \mathbf{s} \quad (10)$$

4.2 Space of majorants and hyperbolic space

The group $\text{SL}(2, \mathbb{C})$ acts on the set of hermitian matrices with given determinant via $[\sigma]T = \sigma T \bar{\sigma}^t$. Every hermitian matrix has the form $T = \begin{pmatrix} a & \beta \\ \bar{\beta} & c \end{pmatrix}$ with $a, c \in \mathbb{R}$ and $\beta = b_1 + ib_2 \in \mathbb{C}$. Under the identification

$$T = \begin{pmatrix} a & \beta \\ \bar{\beta} & c \end{pmatrix} \rightarrow \lambda_T = (a, 2b_1, 2b_2, c) \quad (11)$$

we have

$$4 \det T = \Omega(\lambda_T).$$

This gives rise to an isomorphism of groups

$$\begin{aligned} \text{SL}(2, \mathbb{C})/\{\pm I\} &\rightarrow \text{SO}^+(\Omega) = D^{-1}\text{SO}^+(1, 3)D \\ \sigma &\rightarrow D^{-1}S_\sigma D \end{aligned}$$

satisfying

$$D^{-1}S_\sigma D \lambda_T = \lambda_{[\sigma]T}. \quad (12)$$

Here the “plus” shall denote the identity component of the orthogonal group.

Now to each $w = u + jv \in \mathbb{H}^3$ we associate the matrix $\omega_w = \begin{pmatrix} \sqrt{v} & u/\sqrt{v} \\ 0 & 1/\sqrt{v} \end{pmatrix}$ which in turn corresponds to an element $D^{-1}S_w D \in SO^+(\Omega)$, where we write for short S_w instead of S_{ω_w} .

At least we map w to a majorant

$$w \rightarrow \Omega_w^+ = D^t S_{-\frac{1}{w}}^{-t} S_{-\frac{1}{w}}^{-1} D \quad (13)$$

(where we write A^{-t} for $(A^{-1})^t$) and define

$$\Theta(\mathbf{r}, z, w) := \Theta(\mathbf{r}, z, \Omega_w^+).$$

We compute Ω_w^+ in terms of T and W , see also [17], Lemma 4.1

Lemma 4.1. *Let $w = u + iv \in \mathbb{H}$, $\omega_w = \begin{pmatrix} \sqrt{v} & u/\sqrt{v} \\ 0 & 1/\sqrt{v} \end{pmatrix}$ and $W = \omega_w \overline{\omega_w}^t$. Then for any hermitian matrix T we obtain*

$$\Omega_w^+(\lambda_T) = 2\text{tr}^2((T\overline{W})) - 4 \det(T).$$

Proof. Define $\tau(x_1, x_2, x_3, x_4) := \begin{pmatrix} x_1 & (x_2 + ix_3)/2 \\ (x_2 - ix_3)/2 & x_4 \end{pmatrix}$. So τ is the inverse map for $T \rightarrow \lambda_T$ and the compatibility condition (12) implies

$$D^{-1}S_{-\frac{1}{w}}^{-1}D = \tau^{-1}[\omega_{-\frac{1}{w}}^{-1}]\tau. \quad (14)$$

Further $\Omega_w^+(\lambda) = \lambda^t D^t S_{-\frac{1}{w}}^{-t} S_{-\frac{1}{w}}^{-1} D \lambda = \|S_{-\frac{1}{w}}^{-1} D \lambda\|^2$ (with the usual euclidean vector norm $\|\cdot\|$) which by Eq.(14) yields

$$\Omega_w^+(\lambda_T) = \|D\tau^{-1}[\omega_{-\frac{1}{w}}^{-1}]\tau(\lambda_T)\|^2 = \|D\tau^{-1}[\omega_{-\frac{1}{w}}^{-1}]T\|^2.$$

A straightforward computation shows, that for any hermitian matrix A

$$\|D\tau^{-1}A\|^2 = 2\text{tr}(A^2) = \text{tr}^2(A) - 2 \det(A).$$

Another direct calculation gives

$$\text{tr}([\omega_{-\frac{1}{w}}^{-1}]T) = \text{tr}(T\overline{W})$$

and the proof is complete since $\det W = 1$. □

As a consequence of this lemma we find

$$\Theta(0, z, w) = y \sum_{T \in L} e(4x \det T) e^{-2\pi y [T, w]}, \quad (15)$$

where we write for short $[T, w]$ for $2\text{tr}^2(TW) - 4\det(TW)$. To see this, one should notice as above, that for hermitian matrices T, W

$$\text{tr}(T\bar{W}) = \text{tr}(\bar{T}W)$$

and T, \bar{T} are both contained in L .

Furthermore for $\gamma \in \text{SL}(2, \mathbb{R})$ we obtain by help of lemma ??

$$[[\bar{\gamma}^t]T, w] = [T, \gamma w]$$

and since $\text{SL}(2, \mathcal{O})$ acts on $L_n = \{T \in L : 4\det T = n\}$ this implies

$$\Theta(0, z, \gamma w) = \Theta(0, z, w) \quad (16)$$

for $\gamma \in \text{SL}(2, \mathcal{O})$.

4.3 Kojima's plus space and the theta multiplier system

4.3.1 Vector valued new forms

As in [20] we also associate to $f(z) \in S_{k-1}^*$, where k is a natural number divisible by 4 a vector valued function which is automorphic under the action of $\text{SL}(2, \mathbb{Z})$.

If $f(z) = \sum_{l \in \mathbb{Z}} B(l, y)e(lx) \in S_{k-1}^*$ and $\alpha \in \mathcal{D}^{-1}$, put

$$f_\alpha(z) = \frac{-2iy^{\frac{k}{2}-\frac{1}{2}}}{\chi_K(4\mathcal{N}(\alpha)) + 1} \sum_{l \equiv -4\mathcal{N}(\alpha) \pmod{4}} B(l, y/4)e(lx/4), \quad z = u + iv.$$

Fix a set of representatives for $\mathcal{D}^{-1}/\mathcal{O}$ by $\alpha_1 = 0, \alpha_2 = 1/2, \alpha_3 = i/2, \alpha_4 = (1+i)/2$ and define the vector

$$\mathbf{f}(z) = (f_{\alpha_1}, \dots, f_{\alpha_4})^t.$$

As in [20] we find

$$f_\alpha\left(-\frac{1}{z}\right) = j_{k-1}\left(\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, z\right) \frac{i}{2} \sum_{\beta \in \mathcal{D}^{-1}/\mathcal{O}} e(\alpha\bar{\beta} + \beta\bar{\alpha}) f_\beta(z) \quad (17)$$

and $f_\beta(z+1) = e(-\mathcal{N}(\beta))f_\beta(z)$.

We state this result as

Lemma 4.2. *Let k be a natural number divisible by 4 and $f \in S_{k-1}^*$ and $\mathbf{f}(z)$ defined as above. Then we have*

$$\mathbf{f}\left(\frac{-1}{z}\right) = j_{k-1}\left(\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, z\right) \mathcal{U}\left(\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\right) \mathbf{f}(z)$$

and

$$\mathbf{f}(z+1) = \mathcal{U}\left(\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\right) \mathbf{f}(z)$$

with the unitary matrices

$$\mathcal{U}\left(\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\right) = \frac{i}{\sqrt{2}} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix}$$

and

$$\mathcal{U}\left(\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\right) \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \\ 0 & 0 & -i & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}.$$

Since the transformations $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ generate the modular group we can conclude that the vector-valued function \mathbf{f} is automorphic with respect to $\Gamma_0(1)$ and a unitary representation \mathcal{U} of $\Gamma_0(1)$.

We shall see that this representation is a subrepresentation of the theta representation.

4.3.2 Subrepresentation of the theta representation

We can realize \mathcal{U} as a subrepresentation of χ associated to the (standard-)lattice of of signature $(1, 3)$, level 4 and discriminant group $\mathbb{Z}/4\mathbb{Z}$.

For this define we define for fixed $r_b \in \{(u, v) : u, v \in \{0, \frac{1}{2}\}\}$ the theta series

$$\Theta_{u,v}(z, w) := \sum_{r_a, r_c \in \{0, \frac{1}{2}\}} \Theta((r_a, u, v, r_c), z, w),$$

for which we have the transformation formula

Proposition 4.1. *i)*

$$\Theta_{u,v}\left(\frac{-1}{z}, w\right) = \frac{i}{2} j_{-1}\left(\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, z\right) \sum_{\mu, \nu \in \{0, \frac{1}{2}\}} e(2u\mu + 2v\nu) \Theta_{\mu, \nu}(z, w),$$

$$ii) \quad \Theta_{u,v}(z + 1, w) = e(-(u^2 + v^2)) \Theta_{u,v}(z, w),$$

Proof. The above identities follow by simple calculations from (8)-(10). One should observe, that

$$\sum_{r_a, r_c \in \{0, \frac{1}{2}\}} e(-4s_c r_a - 4s_a r_c + 2s_0 r_0 + 2s_1 r_1) = \begin{cases} 4e(2s_0 r_0 + 2s_1 r_1), & s_a, s_c \in \{0, \frac{1}{2}\} \\ 0 & \text{else.} \end{cases}$$

□

There is a map $\mathbf{D} \rightarrow \mathcal{D}^{-1}/\mathcal{O}$, $(u, v) \rightarrow u + iv$. Under this map (u, v) correspond to $\alpha_i \in \mathcal{D}^{-1}/\mathcal{O}$ chosen above and we also use the notation $\Theta_{\alpha_i}(z, w)$.

Next introduce the vector-valued function

$$\Theta_{\mathbf{red}}(z, w) = (\Theta_{\alpha_1}(z, w), \dots, \Theta_{\alpha_4}(z, w))^t$$

then the above transformation formula can be written as

$$\Theta_{\mathbf{red}}\left(\frac{-1}{z}, w\right) = j_{-1}\left(\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, z\right)\mathcal{U}\left(\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\right)\Theta_{\mathbf{red}}(z, w)$$

and

$$\Theta_{\mathbf{red}}(z+1, w) = \mathcal{U}\left(\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\right)\Theta_{\mathbf{red}}(z, w)$$

Observe that \mathbf{f} transforms under $\mathrm{SL}(2, \mathbb{Z})$ similar to $\Theta_{\mathbf{red}}$.

Remark. Rewrite the defining equation for $\Theta_{u,v}(z, w)$

$$\begin{aligned} \Theta_{u,v}(z, w) &= \sum_{r_a, r_c \in \{0, \frac{1}{2}\}} \Theta(r_a, u, v, r_c), z, w \\ &= y \sum_{a \in \mathbb{Z}} \sum_{c \in \mathbb{Z}} \sum_{b_1 \equiv 2u(2)} \sum_{b_2 \equiv 2v(2)} e\left(x\Omega\left(\frac{a}{2}, \frac{b_1}{2}, \frac{b_2}{2}, \frac{c}{2}\right) + iy\Omega_w^+\left(\frac{a}{2}, \frac{b_1}{2}, \frac{b_2}{2}, \frac{c}{2}\right)\right). \end{aligned}$$

Now it is obvious that $\Theta_{u,v}(z, w)$ are related to the quadratic form $q(a, b, c) = ac - b_1^2 - b_2^2$ with corresponding

$$\Omega_q = \begin{pmatrix} 0 & 0 & 0 & \frac{1}{2} \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ \frac{1}{2} & 0 & 0 & 0 \end{pmatrix}.$$

It has signature $(1, 3)$, level 4 and discriminant group $\mathbb{Z}/4\mathbb{Z}$. Remark, that the discriminant group of q is represented by the set $\{(0, u, v, 0) : u, v \in \{0, \frac{1}{2}\}\}$.

$\Theta(0, z, w)$ has a Fourier expansion

$$\Theta(0, z, w) = y \sum_{n \in \mathbb{Z}} e(nx) \theta_n(y, w)$$

with

$$\theta_n(y, w) = \sum_{T \in L_n} e^{-2\pi y[w, T]}.$$

where $L_n = \{T \in L : 4 \det L = n\}$. This shows that for $n \equiv 3 \pmod{4}$ $L_n = \emptyset$ and hence $\Theta(0, z, w)$ satisfies Kojimas nonvanishing condition for the Fourier coefficients of S_{k-1}^* and it is evident, that

$$\begin{aligned} \Theta_{\alpha_i}(z, w) &= y \sum_{\lambda \in \mathbb{Z}^4, \Omega(\lambda) \equiv -4\mathcal{N}(\alpha_i) \pmod{4}} e(x\Omega(\lambda/2) + iy\Omega^+(\lambda/2)) \\ &= y \sum_{\lambda \in \mathbb{Z}^4, \Omega(\lambda) \equiv -4\mathcal{N}(\alpha_i) \pmod{4}} e\left(\frac{x}{4}\Omega(\lambda) + i\frac{y}{4}\Omega^+(\lambda)\right). \end{aligned}$$

So we see that up to a constant factor the vector valued theta series $\Theta_{\mathbf{red}}(z, w)$ is constructed in the same way from $\Theta(0, z, w)$ as the vector valued forms of the Kojima space. Therefore it does not come to a great surprise that $\Theta_{\mathbf{red}}(z, w)$ behaves similarly under the action of the modular group.

4.4 Rankin-Selberg integral

The next steps are almost identical to the procedure in [17] and we shall omit the details. First we write $\tilde{F}(w, s)$ as a Rankin-Selberg integral where we use the real analytic Eisenstein series for $\mathrm{SL}(2, \mathbb{Z})$ of weight $l \in \mathbb{Z}$ which is given by

$$E_l(z, s) = \sum_{T = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_\infty \backslash \mathrm{SL}(2, \mathbb{Z})} y_T^s e^{-il \arg(cz+d)}.$$

The functional equation

$$E_l(z, s) = \phi_l(s) E_l(z, 1 - s) \quad (18)$$

with

$$\phi_l(s) = \frac{i^{-l} 2^{2-2s} \pi \Gamma(2s-1) \zeta(2s-1)}{\zeta(2s) \Gamma(s - \frac{l}{2}) \Gamma(s + \frac{l}{2})}$$

is well known and we put

for $w \in \mathbb{H}^3$, $f \in S_{k-1}^*$ and \mathbf{f} as introduced above

$$\mathcal{S}_f(w, s) := \int_{\Gamma_0(1) \backslash \mathbb{H}} \langle \mathbf{f}(z), \Theta_{red}(z, w) \rangle E_{-k}(z, s) \frac{dx dy}{y^2}, \quad (19)$$

with $\langle \mathbf{g}, \mathbf{h} \rangle = \sum_{i=1}^2 g_i \overline{h_i}$.

4.5 The main theorem

Our main result is

Theorem 4.1. *Let F be as in theorem 2.1, and W a hermitian positive definite matrix with $\det W = 1$, $w = \phi(W)$. Then F is a unimodular convergent Fourier series as in (2) where the Fourier coefficients are of at most polynomial growth and $A(T) = 0$, if $\det T = 0$. Furthermore the partial Mellin transform $\tilde{F}(w, s)$ has the following properties:*

i) *For all $s \in \mathbb{C}$ with $\Re s$ sufficiently large, we have as an identity of meromorphic functions*

$$\tilde{F}(w, s) = \frac{1}{2} \pi^{-\frac{1}{2}-s} \Gamma(s + \frac{1}{2}) \zeta(2s - k + 1) \mathcal{S}_f(w, s - \frac{k-1}{2}).$$

ii) *The Mellin transform $\tilde{F}(w, s)$ extends to an entire function in s and it satisfies for all $W > 0$ the functional equation*

$$\tilde{F}(w, k - s) = \tilde{F}(w, s).$$

iii) *$\tilde{F}(w, s)$ tends to zero as $\Im s \rightarrow \pm\infty$ uniformly in every vertical strip.*

Proof. This is completely analogous to [17] □

All assumptions of Theorem 3.1 are fulfilled for the Saito-Kurokawa lift and therefore Theorem 2.1 is proven. □

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