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Vector-valued Cusp Forms and Orthogonal Modular Forms

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Zusammenfassung / Abstract:

The aim of this paper is to give a short proof of the Saito-Kurokawa lift for orthogonal modular forms along the lines we gave in two earlier papers. The proof uses a converse theorem as was stated by Imai for Siegel modular forms, yet avoiding the framework of spectral analysis.

Instead we are able to write the partial Mellin transform of the Saito-Kurokawa lift as a Rankin-Selberg integral of the theta lift of f twisted by an Eisenstein series. The functional equation of the Eisenstein series then implies the desired functional equation for the partial Mellin transform which in turn proves the lift to be an orthogonal modular form.

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1 Introduction

There is a well-known correspondence between classical modular forms contained in the Kohnen space and the Maass Spezialschar of Siegel modular forms, called the Saito-Kurokawa-Lift. Kojima [11] and Krieg [8] found an Hermitian anlogue of this lifting. By the work of Maaß, Andrianov and Zagier the Saito-Kurokawa lift was produced as a compostion of two liftings involving Jacobi forms, see [3], the Hermitian analogue was constructed in a similar way. In [18] Sugano generalized this lifting from Jacobi forms to holomorphic cusp forms on $\mathrm{SO}(2, m+2)$. Note that $m = 1$ refers to Siegel- and $m = 2$ to Hermitian modular forms.

A different proof for the Saito-Kurokawa lift for Siegel modular forms was given by Duke and Imamoglu in [2] using a converse theorem of Imai [7]. In [14] we applied the method of Duke and Imamoglu to the Hermitian case.

Recently we gave simpler proofs for the Saito-Kurokawa lift, see [12], [13], where we may skip the analysis of the spectral Koecher-Maass series. The crucial point in our proof is the observation, that the partial Mellin transform of the Saito-Kurokawa-lift of f coincides with a theta lift of f matched with an Eisenstein series, which can easily be evaluated by applying the Rankin Selberg method.

The purpose of the present paper is to generalize this proof for forms on orthogonal groups.

2 Orthogonal modular forms

Siegel modular forms can be regarded as modular forms on $\mathrm{SO}(2, 3)$, the Hermitian case corresponds to $\mathrm{SO}(2, 4)$. Here we shall consider forms on $\mathrm{SO}(2, m + 2)$.

2.1 The tube domain

Let S_1 and S_2 denote the bilinear forms over the reals given by the matrices

$$S_1 = \begin{pmatrix} & & 1 \\ & -2E_m & \\ 1 & & \end{pmatrix}$$

with E_m the m-dim. identity matrix and

$$S_2 = \begin{pmatrix} & & 1 \\ & S_1 & \\ 1 & & \end{pmatrix}$$

and corresponding quadratic forms $Q_i(z) = \frac{1}{2} z^t S_i z =: S_i[z]$. Q_2 has signature $(2, m + 2)$ and Q_1 has signature $(1, m+1)$, the corresponding orthogonal groups $\mathrm{O}(Q_i)$ are therefore isomorphic to $\mathrm{O}(1, m + 1)$ and $\mathrm{O}(2, m + 2)$, respectively. Write $Q_2 = Q_2^+ \perp Q_2^-$ for the orthogonal decomposition of Q_2 into its positive and negative definite parts.

Due to the natural injection

$$h \to \begin{pmatrix} 1 & & \\ & h & \\ & & 1 \end{pmatrix}$$

we consider $\mathrm{O}(Q_1)$ as subgroup of $\mathrm{O}(Q_2)$.

There is a symmetric space associated to $\mathrm{O}(Q_2,\mathbb{R})$. We are going to work with the so-called tube domain model of this space. The tube domain is defined by

$$\mathcal{H}=\{\mathcal{Z}=(z_1,w,z_2)^t\in\mathbb{C}^{m+2}:\Im z_1,\Im z_2>0,w\in\mathbb{C}^m,Q_1(\Im\mathcal{Z})>0\}.$$

In order to understand the action of the orthogonal group on this space one may regard the the zero space $\mathcal{N}=\{(a,\mathcal{Z},b)^t\in\mathbb{C}^{m+4}:Q_2((a,\mathcal{Z},b)^t)=0\}$ on which $\mathrm{O}(Q_2,\mathbb{R})$ acts by matrix multiplication. The condition $Q_2((a,\mathcal{Z},b)^t)=0$ then is equivalent to $ab=-Q_1(\mathcal{Z})$.

Now one can show, that for $\mathcal{Z}\in\mathcal{H}$ and $g\in\mathrm{O}(Q_2,\mathbb{R})$ matrix multiplication gives

$$g\cdot\begin{pmatrix}-Q_1(\mathcal{Z})\\ \mathcal{Z}\\ 1\end{pmatrix}=b(g,\mathcal{Z})\begin{pmatrix}-Q_1(g\langle\mathcal{Z}\rangle)\\ g\langle\mathcal{Z}\rangle\\ 1\end{pmatrix}$$

with some uniquely determined tuple $(b(g,\mathcal{Z}),g\langle\mathcal{Z}\rangle)$.

The factor $b(g,\mathcal{Z})$ is an automorphy factor, it satisfies the chain rule $b(gh,\mathcal{Z})=b(g,h\langle\mathcal{Z}\rangle)(b(h,\mathcal{Z})$.

The map $g:\mathcal{Z}\to g\langle\mathcal{Z}\rangle$ induces a transitive action of the real orthogonal group $\mathrm{O}(Q_2,\mathbb{R})$ on $\mathcal{H}\cup-\mathcal{H}$ as a group of biholomorphic automorphisms in such a way, that either g restricts to an automorphism on $\mathcal{H}$ and $-\mathcal{H}$ or it interchanges the two components. We define $\mathrm{O}(Q_2,\mathbb{R})^+$ to be the subgroup of those automorphisms stabilizing $\mathcal{H}$.

The stabilizer group K_0 of $\mathcal{Z}_0=(i,0,..,0,i)^t$ in $\mathrm{O}(Q_2,\mathbb{R})^+$ is a maximal compact subgroup and so we end up with a symmetric space as stated above.

The injection $\sigma:h\to\begin{pmatrix}1&&\\&h&\\&&1\end{pmatrix}$ provides us with a subgroup $\sigma(\mathrm{O}(Q_1,\mathbb{R}))$ of $\mathrm{O}(Q_2,\mathbb{R})$ and we define

$$\mathrm{O}^+(Q_1,\mathbb{R}):=\{h\in\mathrm{O}(Q_1,\mathbb{R}):\sigma(h)\in\mathrm{O}^+(Q_2,\mathbb{R})\}.$$

For $h\in\mathrm{O}(Q_1,\mathbb{R})$ we have

$$\begin{pmatrix}1&&\\&h&\\&&1\end{pmatrix}\begin{pmatrix}-Q_1(\mathcal{Z})\\ \mathcal{Z}\\ 1\end{pmatrix}=\begin{pmatrix}-Q_1(\mathcal{Z})\\ h\mathcal{Z}\\ 1\end{pmatrix}=\begin{pmatrix}-Q_1(h\mathcal{Z})\\ h\mathcal{Z}\\ 1\end{pmatrix}. \tag{1}$$

Therefore $h\in\mathrm{O}^+(Q_1,\mathbb{R})$ if the matrix multiplication $\mathcal{Z}\to h\mathcal{Z}$ stabilizes $\mathcal{H}$. In this case $\sigma(h)\langle\mathcal{Z}\rangle=h\mathcal{Z}$.

One finds, that $\sigma(\mathrm{SO}^+(Q_1,\mathbb{R}))$ acts transitively on the subspace of purely imaginary elements $i\mathcal{Y}\in\mathcal{H}$ with $Q_1(\mathcal{Y})=1$ (so $Q_1(i\mathcal{Y})=-1$) which corresponds to matrix multiplication on

$$\mathbb{H}_{m+1}=\{\mathcal{Y}=(y_1,v,y_2)^t\in\mathbb{R}^{m+2}:y_1,y_2>0,v\in\mathbb{R}^m,Q_1(\mathcal{Y})=1\}.$$

To be concrete, let $\mathcal{Y}=((1+v^tv)y^{-1},v,y)^t\in\mathbb{H}_{m+1}$. The matrix

$$S_{\mathcal{Y}}=\begin{pmatrix}y^{-1}&2vy^{-1}&v^tvy^{-1}\\ &E_m&v\\ &&y\end{pmatrix} \tag{2}$$

is in $\mathrm{SO}^+(Q_1)$ and maps the element $e^t := (1,0,...,0,1) \in \mathbb{H}_{m+1}$ to $S_{\mathcal{Y}}e = \mathcal{Y}$.

The stabilizer group $K_0 \cap \mathrm{SO}^+(Q_1)$ is in $\mathrm{SO}^+(Q_1,\mathbb{R})$ and so $\mathbb{H}_{m+1}$ corresponds to the symmetric space $\mathrm{SO}^+(Q_1,\mathbb{R})/K_0$ and is therefore a model for $m+1$-dimensional hyperbolic space.

2.2 Modular forms

We define a subgroup Γ of $\mathrm{O}(Q_2,\mathbb{R})^+ \cap \mathrm{GL}_{m+4}(\mathbb{Z})$ generated by the elements

$$J = \begin{pmatrix} & & & & -1 \\ & & & -1 & \\ & & E_m & & \\ & -1 & & & \\ -1 & & & & \end{pmatrix} \quad n(\zeta) = \begin{pmatrix} 1 & -\zeta^t S_1 & -Q_1(\zeta) \\ & E_m & \zeta \\ & & 1 \end{pmatrix},\ \zeta \in \mathbb{Z}^m$$

and

$$\begin{pmatrix} 1 & & \\ & A & \\ & & 1 \end{pmatrix}, \quad A \in \mathrm{O}^+(Q_1,\mathbb{R}) \cap \mathrm{GL}_{m+2}(\mathbb{Z})$$

We define holomorphic modular forms for Γ on the tube domain $\mathcal{H}$.

Definition 2.1. *A holomorphic function F on $\mathcal{H}$ is a holomorphic modular form of weight $k \in \mathbb{N}$ with respect to Γ if*

$$F(\gamma\langle\mathcal{Z}\rangle) = b^k(\gamma,\mathcal{Z})F(\mathcal{Z}) \tag{3}$$

for all $\gamma \in \Gamma$. This space is denoted by $M_k(\Gamma)$.

Note, that

$$n(\zeta)\tilde{\mathcal{Z}} = n(\zeta)\begin{pmatrix} -Q_1(\mathcal{Z}) \\ \mathcal{Z} \\ 1 \end{pmatrix} = \begin{pmatrix} -Q_1(\mathcal{Z}) - \zeta^t S_1 \mathcal{Z} - Q_1(\zeta) \\ \mathcal{Z}+\zeta \\ 1 \end{pmatrix} = \begin{pmatrix} -Q_1(\mathcal{Z}+\zeta) \\ \mathcal{Z}+\zeta \\ 1 \end{pmatrix}.$$

Therefore the matrix $n(\zeta)$ corresponds to the translation $\mathcal{Z} \to \mathcal{Z}+\zeta$. By virtue of (1) we find

$$F(\gamma\langle\mathcal{Z}\rangle) = F(\langle\mathcal{Z}\rangle). \tag{4}$$

for $\gamma \in \Gamma_1 := \Gamma \cap \sigma(\mathrm{O}^+(Q_1,\mathbb{R}))$

We define L to be the set of vectors $\mathcal{T} = (\lambda,\mu,\nu)^t$ where $\lambda,\nu \in \mathbb{Z}, \mu \in \mathbb{Z}^m$. L^+ refers to those with $\lambda,\nu,Q_1(\mathcal{T}) > 0$.

Now each holomorphic modular form $F(\mathcal{Z})$ admits an absolutely convergent Fourier development

$$F(\mathcal{Z}) = \sum_{\mathcal{T}\in L, Q_1(\mathcal{T})\geq 0} A(\mathcal{T})e(\mathcal{T}^t S_1 \mathcal{Z}). \tag{5}$$

Note, that the Koecher-principle is valid, and therefore we have no summands for $\mathcal{T}$ with $Q_1(\mathcal{T}) < 0$. See e.g. ([9]).

If $\mathcal{T} \in L$ and $Q_1(\mathcal{T}) \geq 0$ then for $U \in \Gamma_1$ also $\mathcal{U}\mathcal{T} \in L$ and $Q_1(\mathcal{U}\mathcal{T}) \geq 0$ and the Fourier coefficients satisfy the unimodularity property

$$A(\mathcal{T}) = A(U\mathcal{T}) \tag{6}$$

for any $U \in \mathrm{O}^+(Q_1) \cap \mathrm{GL}(m+2, \mathbb{Z})$. The latter follows from (4) by virtue of

$$\begin{aligned} F(U\mathcal{Z} &= \sum_{\mathcal{T}\in L,\, Q_1(\mathcal{T})\geq 0} A(\mathcal{T}) e(\mathcal{T}^t S_1 U \mathcal{Z}) \\ &= \sum_{\mathcal{T}\in L,\, Q_1(\mathcal{T})\geq 0} A(\mathcal{T}) e(\mathcal{T}^t U^{-t} S_1 \mathcal{Z}) \\ &= \sum_{\mathcal{T}\in L,\, Q_1(\mathcal{T})\geq 0} A(\mathcal{T}) e((U^{-1}\mathcal{T})^t S_1 \mathcal{Z}) \\ &= \sum_{\mathcal{T}\in L,\, Q_1(\mathcal{T})\geq 0} A(U\mathcal{T}) e(\mathcal{T}^t S_1 \mathcal{Z}). \end{aligned}$$

2.3 Saito-Kurokawa lift

Let $\mathcal{U}$ be the finite dimensional representation of $\mathrm{SL}(2,\mathbb{Z})$ introduced in section 3 and for even $k > \frac{m}{2}$ let $f = (f_\alpha)$ be a vector-valued cusp form from $S(k - m/2, \mathcal{U})$ with Fourier coefficients $a_\alpha(n)$, see section 5 for more details.

The function $F(\mathcal{Z})$ on $\mathcal{H}$ defined by

$$F(\mathcal{Z}) = \sum_{\mathcal{T}\in L} A(\mathcal{T}) e(\mathcal{T}^t S_1 \mathcal{Z}) \tag{7}$$

where $A(\mathcal{T}) = \sum_{d|\varepsilon(\mathcal{T})} d^{k-1} a^*(\frac{4Q_1(\mathcal{T})}{d^2})$ with $\varepsilon(\mathcal{T}) = \max\{q \in \mathbb{N} : \mathcal{T}/q \in L\}$ and

$$a^*(n) = \sum_{\substack{\alpha \\ 4Q_1(\alpha) \equiv n (\mathrm{mod}\, 4)}} a_\alpha(n).$$

is called the Saito Kurokawa lift of f.

The main purpose of this paper is to present a short proof for the following

Theorem 2.1. *Let $f \in S(k - m/2, \mathcal{U})$ and $F(Z)$ is the corresponding Saito-Kurokawa lift. Then $F(Z)$ is an element of $S_k(\Gamma)$.*

For the proof we show that for $\mathcal{Y} \in \mathbb{H}_{m+1}$ the Mellin transform

$$\tilde{F}(i\mathcal{Y}, s) = \int_0^\infty F(iu^{\frac{1}{2}}\mathcal{Y}) u^{s-1} du$$

satisfies a functional equation

$$\tilde{F}(i\mathcal{Y}, s) = \tilde{F}(i\mathcal{Y}, k - s) \tag{8}$$

and then use a converse theorem. The functional equation may result from corresponding equations for the spectral Koecher-Maass series. This was elabourated by Imai in [7] and used by Duke and Imamoglu in [2].

Here we use a shorter argument as we already did in [12], [13]. It results from the observation that

$$\tilde{F}(i\mathcal{Y},s)=\frac{1}{2}\pi^{-\frac{1}{2}-s}\Gamma(s+\frac{1}{2})\zeta(2s-k+1)\mathcal{S}_f(\mathcal{Y},s-\frac{k-1}{2})$$

where $\mathcal{S}_f(\mathcal{Y},s-\frac{k-1}{2})$ is a theta lift of f matched with a real-analytic Eisenstein series, see (21). The analytic properties of the Eisenstein series, especially its functional equation, carry over to $\tilde{F}(i\mathcal{Y},s)$.

We proceed with delivering the details of the argument.

3 Theta series

3.1 Theta series for Q_1

We have a closer look at the quadratic form $Q_1=\frac{1}{2}S_1$.

Lemma 3.1. *Let $\mathcal{T}\in L^+$, $\mathcal{Y}\in\mathbb{H}_{m+1}$. Then*

i) $\mathcal{T}^t S_1\mathcal{Y}>0$

ii) $(\mathcal{T}^t S_1\mathcal{Y})^2=2Q_1^+[S_{\mathcal{Y}}^{-1}][\mathcal{T}]+2Q_1[\mathcal{T}]$ *with* $Q_1^+=\frac{1}{2}\begin{pmatrix}1&&\\&2E_m&\\&&1\end{pmatrix}$.

iii) $Q_1^+[S_{\gamma\mathcal{Y}}^{-1}][\mathcal{T}]=Q_1^+[S_{\mathcal{Y}}^{-1}][\gamma^{-1}\mathcal{T}]$ *for* $\gamma\in\Gamma_1$.

Remark. The matrix $P=Q_1^+[S_{\mathcal{Y}}^{-1}]$ is a majorant for Q_1 in the sense of Siegel, i.e. P is symmetric and positive definit and $PQ_1^{-1}P=Q_1$. One may check this by a simple calculation.

Proof. (i) Let $\mathcal{Y}=((1+v^tv)y^{-1},v,y)$ with $y>0$. Then $e^tS_1\mathcal{Y}=(1+v^tv)y^{-1}+y>0$. Since $\mathcal{T}^0:=\mathcal{T}/\sqrt{Q_1(\mathcal{T})}\in\mathbb{H}_{m+1}$ we can choose $S_{\mathcal{T}^0}\in\mathrm{SO}^+(Q_1;\mathbb{R})$, see (2), such that $S_{\mathcal{T}^0}e=\mathcal{T}^0$. Therefore

$$\mathcal{T}^tS_1\mathcal{Y}=\sqrt{Q_1(\mathcal{T})}e^tS_1S_{\mathcal{T}^0}^{-1}\mathcal{Y}>0$$

since $S_{\mathcal{T}^0}^{-1}\mathcal{Y}\in\mathbb{H}_{m+1}$.

(ii)

$$\begin{aligned}(\mathcal{T}^tS_1\mathcal{Y})^2&=\mathcal{T}^tS_1\mathcal{Y}\mathcal{Y}^tS_1\mathcal{T}=\mathcal{T}^tS_1S_{\mathcal{Y}}ee^tS_{\mathcal{Y}}^tS_1\mathcal{T}\\&=\mathcal{T}^tS_{\mathcal{Y}}^{-t}S_1ee^tS_1S_{\mathcal{Y}}^{-1}\mathcal{T}\\&=\mathcal{T}^tS_{\mathcal{Y}}^{-t}ee^tS_{\mathcal{Y}}^{-1}\mathcal{T}.\end{aligned}$$

The statement follows since $ee^t=S_1^++S_1$.

(iii) This follows from (ii) since $\mathcal{T}^tS_1\gamma\mathcal{Y}=\mathcal{T}^t\gamma^{-t}S_1\mathcal{Y}$ and Γ_1 acts on L^+. □

The theta series associated to the indefinite quadratic form $Q_1=\frac{1}{2}S_1$ is the following series which converges absolutely on $\mathbb{H}_2\times\mathbb{H}_{m+1}$.

Definition 3.1. *Let $z=x+iy\in\mathbb{H}_2$ and $\mathcal{Y}\in\mathbb{H}_{m+1}$.*

$$\Theta_{Q_1}(z,\mathcal{Y}):=y^{(m+2)/4}\sum_{\mathcal{T}\in L}e(zQ_1(\mathcal{T}))e^{-4\pi y(\mathcal{T}^tQ_1\mathcal{Y})^2}.$$

We can write this as

$$\Theta_{Q_1}(z,\mathcal{Y}) = y^{(m+2)/4}\sum_{n\in\mathbb{Z}} e(\frac{n}{4}x) \sum_{\mathcal{T}\in L_n} e^{-4\pi y(\mathcal{T}^t Q_1 \mathcal{Y})^2} \tag{9}$$

with $L_n = \{\mathcal{T}\in L : 4Q_1(\mathcal{T}) = n\}$.

From the definition it is clear, that $\Theta_{Q_1}(z,\gamma\mathcal{Y}) = \Theta_{Q_1}(z,\mathcal{Y})$ for $\gamma\in\Gamma_1$ since Γ_1 acts on L. (It even acts on L_n.)

3.2 Siegel's theta function

We map L to $\mathbb{Z}^{m+2}$ via $\tau : (\lambda,\mu,\nu)^t \to (\lambda, 2\mu, \nu)^t$ then the quadratic form transforms according $Q_1(\mathcal{T}) = \frac{1}{4}\Omega(\tau\mathcal{T})$ with the quadratic form $\Omega = 4\tau^{-t}Q_1\tau^{-1}$ and $Q_1^+[S_{\mathcal{Y}}^{-1}][\mathcal{T}] = \frac{1}{4}\Omega^+[\tau S_{\mathcal{Y}}^{-1}\tau^{-1}][\tau\mathcal{T}]$ with $\Omega^+ = 4\tau^{-t}Q_1^+\tau^{-1}$. So

$$\Omega = \begin{pmatrix} & & 2 \\ & -E_m & \\ 2 & & \end{pmatrix}, \quad \Omega^+ = \begin{pmatrix} 2 & & \\ & E_m & \\ & & 2 \end{pmatrix}.$$

We write for short $\Omega_{\mathcal{Y}}^+ := \Omega^+[\tau S_{\mathcal{Y}}^{-1}\tau^{-1}]$. Then the above theta series associated with Q_1 can be rewritten in the form

$$\Theta_{Q_1}(z,\mathcal{Y}) = y^{(m+2)/4}\sum_{\nu\in\mathbb{Z}^{m+2}} e(\frac{x}{4}\Omega(\nu) + i\frac{y}{4}\Omega_{\mathcal{Y}}^+(\nu)) == y^{(m+2)/4}\sum_{\nu\in\mathbb{Z}^{m+2}} e(x\Omega(\frac{\nu}{2}) + iy\Omega_{\mathcal{Y}}^+(\frac{\nu}{2})). \tag{10}$$

It is thus obvious, that $\Theta_{Q_1}(z,\mathcal{Y})$ corresponds to a theta series according to Siegel's original definition, see e.g. [16].

To be more precise, there is a Siegel's theta function for each element from the discriminant group associated to the bilinear form 2Ω. Namely, let $\Lambda = \mathbb{Z}^{m+2}$ and denote by $\hat{\Lambda}$ the lattice dual for $\mathbb{Z}^{m+2}$ with respect to Ω and by $\mathcal{D} = \hat{\Lambda}/\Lambda$ the discriminant group for 2Ω.

For $\mathbf{r}\in\mathcal{D}$, $z = x+iy\in\mathbb{H}$ the Siegel theta function for the indefinite form Ω is defined by

$$\Theta(\mathbf{r},z,\mathcal{Y}) := y^{\frac{m+2}{4}}\sum_{\nu\equiv\mathbf{r}(\mathrm{mod}\,\Lambda)} e(x\Omega(\nu) + iy\Omega_{\mathcal{Y}}^+(\nu)).$$

We express $\Theta_{Q_1}(z,\mathcal{Y})$ in terms of these theta functions.

$$\Theta_{Q_1}(z,\mathcal{Y}) \;=\; \sum_{\mathbf{r}'}\Theta(\mathbf{r}',z,\mathcal{Y}), \tag{11}$$

where the sum is over all $\mathbf{r}'\in\mathcal{D}$ with $2\mathbf{r}'\in\mathbb{Z}^{m+2}$.

3.3 Growth estimate

From [16], p.81 and p.117 it follows, that for each $\mathcal{Y}$ there is a $\gamma\in\Gamma_1$ depending on $\mathcal{Y}$, such that $\Omega_{\mathcal{Y}}^+[\gamma] = \Omega_{\gamma\mathcal{Y}}^+$ is reduced in the sense of Minkowski and therefore there is a constant $c>0$ depending on $\mathcal{Y}$ for which we have the estimate

$$e^{-2\pi y\Omega_{\mathcal{Y}}^+[\gamma](\nu)} \le e^{-cy\sum_{i=1}^{m+2}\omega_i\nu_i^2}.$$

Here $\omega_i > 0$ are the diagonal elements of $\Omega^+_{\mathcal{Y}}[\gamma]$. Therefore for fixed $\mathcal{Y}$ the theta series is majorized by (if we replace $\mathcal{Y}$ by $\gamma\mathcal{Y}$)

$$y^{(m+2)/4} \prod_{i=1}^{m+2} \sum_{n=0}^{\infty} e^{-yc\omega_i n^2}. \tag{12}$$

3.4 Vector-valued theta functions

Let $V = \mathbb{C}^{|\mathcal{D}|}$ be equipped with a unitary basis $v_{\mathbf{r}}$ which is indexed by $\mathbf{r} \in \mathcal{D}$.

Subject to this basis we build a vector-valued theta function

$$\mathbf{\Theta}(z, \mathcal{Y}) = \sum_{\mathbf{r}} \Theta(\mathbf{r}, z, \mathcal{Y}) v_{\mathbf{r}}.$$

Siegel's theta function can be constructed using the Weil representation as was done by Shintani in [17]. From this it becomes evident, see [17], Proposition 1.6, that for $\gamma \in \mathrm{SL}(2,\mathbb{Z})$ there is a transformation formula

$$\mathbf{\Theta}(\gamma z, \mathcal{Y}) = j_{-m/2}(\gamma, z)\chi(\gamma)\mathbf{\Theta}(z, \mathcal{Y})$$

with the automorphy factor $j_{-m/2}(\gamma, z) = \left(\frac{cz+d}{c\bar{z}+d}\right)^{-m/4}$ and a unitary matrix $\chi(\gamma) = (\chi_{\mathbf{r},\mathbf{s}}(\gamma))_{\mathbf{r},\mathbf{s}\in\mathcal{D}}$.

Lemma 3.2. *For* $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2,\mathbb{Z})$ *one has*

$$\chi_{\mathbf{r},\mathbf{s}}(\gamma) = \begin{cases} e(\frac{m}{8})|\det(2c\Omega)|^{-\frac{1}{2}} \sum_{g\in\Lambda/c\Lambda} e\left(\frac{a\Omega[g+\mathbf{r}]-2\mathbf{s}^t\Omega(g+\mathbf{r})+d\Omega[\mathbf{s}]}{c}\right), & c \neq 0 \\ e(ab\Omega[\mathbf{r}]) & c = 0, a\mathbf{r} = \mathbf{s} \\ 0 & c = 0, a\mathbf{r} \neq \mathbf{s}. \end{cases}$$

□

This was originally shown by Siegel in [16], Hilfssatz 1.

As in [1], [16] we infer from this that the 'Thetanullwert' $\Theta(0, z, \mathcal{Y})$ is an automorphic form for $\Gamma_0(4)$, namely

$$\Theta(0, \gamma z, \mathcal{Y}) = \left(\frac{c}{d}\right)^m \left(\frac{-4}{d}\right)^{\frac{m}{2}} j_{-m/2}(T, z)\Theta(0, z, \mathcal{Y}) \tag{13}$$

for $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(4)$.

3.5 A subrepresentation of χ

We now choose a subrepresentation $\mathcal{U}$ of χ which for $m = 1$ and $m = 2$ corresponds to the multiplier system for the theta decompostion of the first Fourier-Jacobi coeffcient of a Siegel (m=1) or Hermite (m=2) modular function. For this let $D_m = \{0, \frac{1}{2}\}^m$ be the discriminant group for the lattice corresponding to $2E_m$ and define for $\alpha \in D_m$

$$\Theta_\alpha(z, \mathcal{Y}) = \sum_{2r_a, 2r_c \in \mathbb{Z}} \Theta((r_a, \alpha, r_c)), z, \mathcal{Y}).$$

We obtain the transformation formula

Lemma 3.3. *i)*

$$\Theta_\alpha(\frac{-1}{z},\mathcal{Y}) = 2^{-m/2-2}e(\frac{m}{8})j_{-m/2}(\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix},z)\sum_{\beta\in D_m} e(2\alpha^t\beta)\Theta_\beta(z,\mathcal{Y}),$$

ii) $\Theta_\alpha(z+1,\mathcal{Y}) = e(-\alpha^t\alpha)\Theta_\alpha(z,\mathcal{Y})$.

Proof. We compute the values of χ for the generators $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ and $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ of the modular group. Write $\mathbf{r} = \begin{pmatrix} r_a \\ r_b \\ r_c \end{pmatrix}$ and $\mathbf{s} = \begin{pmatrix} s_a \\ s_b \\ s_c \end{pmatrix}$ with $r_b, s_b \in D_m$.

i)

$$\chi_{\mathbf{r},\mathbf{s}}(\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}) = \frac{1}{2^{m/2+2}}e(\frac{m}{8})e(-4s_cr_a - 4s_ar_c + 2r_b^ts_b) \tag{14}$$

,

ii)

$$\chi_{\mathbf{r},\mathbf{r}}(\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}) = e(4r_ar_c - r_b^tr_b) \tag{15}$$

,

iii)

$$\chi_{\mathbf{r},\mathbf{s}}(\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}) = 0, \quad \mathbf{r}\neq\mathbf{s} \tag{16}$$

The formulas stated by the Lemma follow by simple calculations from (14)-(16). Just remark, that

$$\sum_{r_a,r_c\in\{0,\frac{1}{2}\}} e(-4s_cr_a - 4s_ar_c + 2r_b^ts_b) = \begin{cases} 4e(2r_b^ts_b), & s_a,s_c\in\{0,\frac{1}{2}\} \\ 0 & \text{else.} \end{cases}$$

□

Due to the decomposition of the disciminant group $\mathcal{D} = \mathcal{A}\times D_m$ with $\mathcal{A} = \{0,\frac{1}{4},\frac{1}{2},\frac{3}{4}\}^2$, the representation space $V = \mathbb{C}^{|\mathcal{D}|}$ for χ can be written as $\mathbb{C}^{16}\otimes\mathbb{C}^{2^m}$ with respect to a suitably chosen basis. If V' is the direct summand of V related to the subgroup $\mathcal{D}' = \{\mathbf{r}\in\mathcal{D} : 2\mathbf{r}\in\mathbb{Z}^{m+2}\}$ then V' can be identified with $\mathbb{C}^4\otimes\mathbb{C}^{2^m}$. Let $\pi : V\to V'$ denote the projection and $\sigma : \sum_i\lambda_iv_i \to \sum_i\lambda_i$ is the trace operator on V'.

From the above lemma we deduce that $(\sigma\circ\pi)\otimes E_m$ intertwines with χ and therefore determines a subrepresentation $\mathcal{U}$ on $\mathbb{C}^{2^m}$.

The vector valued function

$$\mathbf{\Theta}_{red}(z,\mathcal{Y}) := (\Theta_\alpha(z,\mathcal{Y})), \quad \alpha\in D_m, \tag{17}$$

is a vector valued modular form w.r.t. $\mathcal{U}$ of weight $-m/2$:

Lemma 3.4. *For $T\in\mathrm{SL}(2,\mathbb{Z})$ we have*

$$\mathbf{\Theta}_{red}(Tz,\mathcal{Y}) = j_{-m/2}(T,z)\mathcal{U}(T)\mathbf{\Theta}_{red}(z,\mathcal{Y}).$$

□

Remark. For $m = 1, 2$ the representation $\mathcal{U}$ coincides with the representation for the vector-valued theta series which comes from the theta decomposition of the corresponding Jacobi forms, cf. [3],[9], [14], [12], [13].

Moreover, in order to establish the Saito-Kuraokawa lift for forms from the Kohnen plus-space $f \in S^+_{k-\frac{1}{2}}$ (if $m = 1$) or the Kojima plus-space $f \in S^*_{k-1}$ (if $m = 2$) one is going to construct vector valued cusp forms in the following way:

Let $f(z) = \sum_{n>0} a(n)e(nz)$ be the Fourier expansion for f then put for $\alpha \in D_m$

$$f_\alpha(z) = \frac{1}{l_\alpha} \sum_{n \equiv 4Q_1(0,\alpha,0) (\mathrm{mod}\, 4)} a(n)e(nz/4),$$

where $l_\alpha = |\{\beta \in D_m : Q_1(0, \beta, 0) = Q_1(0, \alpha, 0)\}|$.

Then (f_α) is a vector valued automorphic form for the representation $\mathcal{U}$. See also [2].

One should observe, that this construction applied to the Nullwert gives our $\mathbf{\Theta}_{red}$.

4 Unimodular invariant Fourier series

Now let us assume, that an absolutely convergent Fourier series

$$F(\mathcal{Z}) = F(\tau, w, z) = \sum_{\mathcal{T} \in L, Q_1(\mathcal{T}) \geq 0} A(\mathcal{T})e(\mathcal{T}^t S_1 \mathcal{Z})$$

on $\mathcal{H}$ with the property (6) is given, not necessarily coming from a modular form. Such a Fourier series we call unimodular invariant.

The unimodular invariance implies

$$F(U\mathcal{Z}) = F(\mathcal{Z}).$$

Remark again, that $e(\mathcal{T}^t S_1((U\mathcal{Z})) = e(\mathcal{T}^t U^{-t} S_1 \mathcal{Z}) = e((U^{-1}\mathcal{T})^t S_1 \mathcal{Z})$ and $\mathcal{T} \in L$ is equivalent to $U^{-1}\mathcal{T} \in L$.

We further assume, that for $\mathcal{Y} \in \mathbb{H}_{m+1}$ the partial Mellin transform

$$\tilde{F}(i\mathcal{Y}, s) = \int_0^\infty F(iu^{\frac{1}{2}}\mathcal{Y})u^{s-1}du \tag{18}$$

exists for $\Re(s)$ sufficiently large and Mellin inversion can be applied.

We remind the reader that Mellin inversion can be applied if F is continuous and the integral is absolutely convergent in some strip $a < \Re(s) < b$

For our proof of the Saito-Kurokawa lift we shall employ the following converse theorem which for the case of Siegel modular forms can be found in [7].

Proposition 4.1. *Let F be defined by an absolutely convergent unimodular Fourier series as in (5) where the Fourier coefficients are of at most polynomial growth $A(\mathcal{T}) \ll Q_1(\mathcal{T})^a$, with some $a > 0$, and $A(\mathcal{T}) = 0$, if $Q_1(\mathcal{T}) = 0$. If for each $\mathcal{Y} \in \mathbb{H}_{m+1}$ the partial Mellin transform $\tilde{F}(\mathcal{Y}, s)$ fulfills the following three conditions:*

i) it is entire as a function of s

ii) it tends to zero as $\Im s \to \pm\infty$ uniformly in every vertical strip *iii) it satisfies the functional equation*

$$\tilde{F}(\mathcal{Y}, s) = (-1)^k \tilde{F}(\mathcal{Y}, k-s) \tag{19}$$

then

$F(\mathcal{Z})$ is a modular form of weight k.

Proof.

We have to check the transformation formula (6) for the generators of the group Γ. By our assumptions on F we merely have to show that

$$F(J\mathcal{Z}) = Q_1(\mathcal{Z})^k F(\mathcal{Z}). \tag{20}$$

Following [10], Lemma 1.6, p.48, it is enough to show the transformation property for the imaginary parts $i\mathcal{W} \in \mathcal{H}$. Each $i\mathcal{W} \in \mathcal{H}$ can be written as $t^{\frac{1}{2}}\mathcal{Y}$ with $\mathcal{Y} \in \mathbb{H}_{m+1}$ and suitable $t > 0$. Then (23) becomes

$$F(Jit^{\frac{1}{2}}\mathcal{Y}) = (-t)^k F(it^{\frac{1}{2}}\mathcal{Y}).$$

Further

$$J\langle it^{\frac{1}{2}}\mathcal{Y}\rangle = it^{-\frac{1}{2}}V\mathcal{Y}$$

where

$$V = \begin{pmatrix} & & 1 \\ & -E_m & \\ 1 & & \end{pmatrix}.$$

Note that $V \in \mathrm{O}^+(Q_1, \mathbb{R}) \cap \mathrm{GL}(m+2, \mathbb{Z})$ and unimodularity implies $F(it^{\frac{1}{2}}\mathcal{Y}) = F(it^{\frac{1}{2}}V\mathcal{Y})$. (Our V is $-V$ in ([9]).

Now we use the Mellin inversion formula, which is valid for $\Re s > c$, for some suitable $c \in \mathbb{R}$ and obtain

$$\begin{aligned}
F(it^{\frac{1}{2}}\mathcal{Y}) &= \frac{1}{2\pi i}\int_{\Re s = c} \tilde{F}(\mathcal{Y}, s)t^{-s}ds \\
&= \frac{(-1)^k}{2\pi i}\int_{\Re s = c} \tilde{F}(\mathcal{Y}, k-s)t^{-s}ds \\
&= \frac{(-1)^k}{2\pi i}\int_{\Re s = k-c} \tilde{F}(\mathcal{Y}, s)t^{-k+s}ds \\
&= \frac{(-1)^k t^{-k}}{2\pi i}\int_{\Re s = k-c} \tilde{F}(\mathcal{Y}, s)\left(\frac{1}{t}\right)^{-s}ds \\
&= (-t)^{-k}F(it^{-\frac{1}{2}}\mathcal{Y}).
\end{aligned}$$

For the last equality we move the path of integration back to $\Re s = c$, which is possible because $\tilde{F}(\mathcal{Y}, s)$ is entire and tends to zero as $\Im s \to \pm\infty$ uniformly in every vertical strip.

By unimodularity the last term equals $(-t)^{-k}F(it^{-\frac{1}{2}}V\mathcal{Y})$ and we get the desired result. □

5 Proof of the theorem

Definition 5.1. *By $S(k-\frac{m}{2},\mathcal{U})$ we denote the space of vector-valued cusp forms of weight $k-\frac{m}{2}$ that transform according to the representation $\mathcal{U}$ of* $\mathrm{SL}(2,\mathbb{Z})$

$$\mathbf{f}(\frac{az+b}{cz+d}) = (cz+d)^{k-\frac{m}{2}}\mathcal{U}\left(\left(\begin{array}{cc} a & b \\ c & d \end{array}\right)\right)\mathbf{f}(z)$$

for $\left(\begin{array}{cc} a & b \\ c & d \end{array}\right) \in \mathrm{SL}(2,\mathbb{Z})$.

See the above remark how these spaces correspond for $m = 1,2$ to the Kohnen plus space $S^+_{k-\frac{1}{2}}$ or Kojima's plus space S^*_{k-1}, respectively.

Any $f = (f_\alpha) \in S(k-m/2,\mathcal{U})$ has Fourier expansions of its components

$$f_\alpha(z) = \sum_{n_\alpha>0} a_\alpha(n)e(n_\alpha z)$$

with $n_\alpha := n + Q_1(0,\alpha,0)$. The occurence of n_α reflects the fact that $f_\alpha(z+1) = e(Q_1(0,\alpha,0))f_\alpha(z)$.

We defined the Saito-Kurokawa lift of f to be the function

$$F(\mathcal{Z}) = \sum_{\mathcal{T}\in L}\left(\sum_{d|\varepsilon(\mathcal{T})} d^{k-1}a^*(\frac{4Q_1(\mathcal{T})}{d^2})\right)e(\mathcal{T}^tS_1\mathcal{Z})$$

The proof of the theorem is complete if we can verify (i)-(iii) of Proposition 4.1. This will be accomplished by the Proposition below.

We write $\tilde{F}(\mathcal{Y},s)$ as a Rankin-Selberg integral and exploit analytic properties of the real analytic Eisenstein series for $\mathrm{SL}(2,\mathbb{Z})$ of weight $l\in\mathbb{Z}$ which is given by

$$E_l(z,s) = \sum_{T=\left(\begin{array}{cc} a & b \\ c & d \end{array}\right)\in\Gamma_\infty\backslash\mathrm{SL}(2,\mathbb{Z})} y_T^s e^{-il\arg(cz+d)}.$$

As a function of s the Eisenstein series $E_l(z,s)$ has an analytic continuation to the whole complex plane. If $l\neq 0$ then $\zeta(2s)E_l(z,s)$ is an entire function in s and it satisfies the functional equation

$$E_l(z,s) = \phi_l(s)E_l(z,1-s)$$

with

$$\phi_l(s) = \frac{i^{-l}2^{2-2s}\pi\Gamma(2s-1)\zeta(2s-1)}{\zeta(2s)\Gamma(s-\frac{l}{2})\Gamma(s+\frac{l}{2})}.$$

This is well known.

Now for $f = (f_\alpha)\in S(k-m/2,\mathcal{U})$ and $\mathcal{Y}\in\mathbb{H}_{m+1}$ we define

$$\mathcal{S}_f(\mathcal{Y},s) := \int_{\mathrm{SL}(2,\mathbb{Z})\backslash\mathbb{H}} y^{\frac{k}{2}-\frac{m}{4}} f(z)^t\overline{\mathbf{\Theta}_{red}(z,\mathcal{Y})}E_{-k}(z,s)\frac{dxdy}{y^2}. \qquad (21)$$

The integral exists for any s which is not a pole of the Eisenstein series, if f is a cusp form. Use the estimate (12) for the theta series. Clearly the integrand is $\mathrm{SL}(2,\mathbb{Z})$-invariant. Now we can show the announced Proposition.

Proposition 5.1. *Let F be the Saito-Kurokawa lift of f as above. Then F is defined by a unimodular convergent Fourier series as in (5) where the Fourier coefficients are of at most polynomial growth and $A(\mathcal{T}) = 0$, if $Q_1(\mathcal{T}) = 0$. Furthermore for $\mathcal{Y} \in \mathbb{H}_{m+1}$ the partial Mellin transform $\tilde{F}(\mathcal{Y}, s)$ has the following properties:*

i) For all $s \in \mathbb{C}$ with $\Re s$ sufficiently large, we have as an identity of meromorphic functions

$$\tilde{F}(\mathcal{Y}, s) = \pi^{-\frac{1}{2}-s}\Gamma(s+\frac{1}{2})\zeta(2s-k+1)\mathcal{S}_f(\mathcal{Y}, s-\frac{k-1}{2}).$$

ii) The Mellin transform $\tilde{F}(\mathcal{Y}, s)$ extends to an entire function in s and it satisfies the functional equation

$$\tilde{F}(\mathcal{Y}, k-s) = \tilde{F}(\mathcal{Y}, s).$$

iii) $\tilde{F}(\mathcal{Y}, s)$ tends to zero as $\Im s \to \pm\infty$ uniformly in every vertical strip.

Proof. Absolute convergence of the defining series is guaranteed from the estimate $A(\mathcal{T}) \ll Q_1^a(\mathcal{T})$ which in turn follows from well known estimates for the Fourier coefficients of cusp forms. Also the condition of unimodularity is fulfilled due to the special form of the Fourier coefficients.

As in [7], p.910f we conclude that the partial Mellin transform exists for $\Re s$ sufficiently large.

The Rankin Selberg integral is evaluated as usual by unfolding the fundamental domain. Using (9)-(11) we obtain

$$\begin{aligned}\mathcal{S}_f(\mathcal{Y}, s-\frac{k-1}{2}) &= \int_0^\infty \int_0^1 f(z)^t\boldsymbol{\Theta}_{red}(z,\mathcal{Y})y^{s-(k-1)/2-2}dxdy \\ &= 2\int_0^\infty \sum_{n>0} a^*(n) \sum_{\mathcal{T}\in L_n\cap L^+} e^{-\pi y(\mathcal{T}^t S_1 \mathcal{Y})^2} y^{s-1}dy \\ &= \frac{2}{\pi^s}\Gamma(s)\sum_{n>0} a^*(n) \sum_{\mathcal{T}\in L_n\cap L^+} \frac{1}{(\mathcal{T}^t S_1\mathcal{Y})^{2s}}.\end{aligned}$$

We used that $\sum_{\mathcal{T}\in L_n} = 2\sum_{\mathcal{T}\in L_n\cap L^+}$ due to the fact, that $(\lambda,\mu,\nu)^t \in L_n$ implies $(-\lambda,\mu,-\nu)^t \in L_n$.

We write as before $\mathcal{T} = \sqrt{Q_1(\mathcal{T})}\mathcal{T}^0$ and arrive at

$$\mathcal{S}_f(\mathcal{Y}, s-\frac{k-1}{2}) = \frac{2^{2s+1}}{(\pi)^s}\Gamma(s)\sum_{n>0}\frac{a^*(n)}{n^s}\sum_{\mathcal{T}\in L_n\cap L^+}\frac{1}{((\mathcal{T}^0)^t S_1\mathcal{Y})^{2s}}. \tag{22}$$

We compare this with the Dirichlet series we obtain when calculating the Mellin transform of F.

$$\begin{aligned}
\int_0^\infty F(iu^{\frac{1}{2}}\mathcal{Y})u^{s-1}du &= 2\sum_{\mathcal{T}\in L^+} A(\mathcal{T})\int_0^\infty e^{-2\pi\sqrt{u}(\mathcal{T}^t S_1\mathcal{Y})}u^{s-1}du \\
&= 4\sum_{\mathcal{T}\in L^+}\frac{A(\mathcal{T})}{(2\pi\mathcal{T}^t S_1\mathcal{Y})^{2s}}\int_0^\infty e^{-v}v^{2s-1}dv \\
&= \frac{4}{\pi^{2s}}\Gamma(2s)\sum_{n>0}\frac{1}{n^s}\sum_{\mathcal{T}\in L_n\cap L^+}\frac{A(\mathcal{T})}{((\mathcal{T}^0)^t S_1\mathcal{Y})^{2s}} \\
&= \frac{4}{\pi^{2s}}\Gamma(2s)\sum_{n>0}\frac{\sum_{d^2|n} d^{k-1}a^*(n/d^2)}{n^s}\sum_{\mathcal{T}/d\in L_{n/d^2}\cap L^+}\frac{1}{((\mathcal{T}^0)^t S_1\mathcal{Y})^{2s}}
\end{aligned}$$

.

For the last equality observe that $d|\varepsilon(\mathcal{T})$ and $\mathcal{T}\in L_n$ iff $\mathcal{T}/d\in L_{n/d^2}$.

If we use the identity for Dirichlet series:

$$\zeta(2s-k+1)\sum_{n>0}\frac{c(n)}{n^s}=\sum_{n>0}\frac{\sum_{d^2|n} d^{k-1}c(n/d^2)}{n^s}$$

and the well known identity for the Gamma-function

$$\frac{\Gamma(2s)}{\Gamma(s)}=\frac{2^{2s-1}}{\sqrt{\pi}}\Gamma(s+\frac{1}{2}), \tag{23}$$

the statement of (i) follows from (22) for all s with $\Re s$ sufficiently large.

(ii) Since f is a cusp form the Rankin Selberg integral gives a meromorphic continuation for $\tilde{F}(\mathcal{Y},s)$ as function of s to the whole complex plane.

It is entire since $\zeta(2s-k+1)E_{-k}(z,s-k/2+1/2)$ is an entire function for even positive weight and the poles of $\Gamma(s+1/2)$ at $s=-1/2-n$ for $n\in\mathbb{N}$ are annihilated by the zeroes of the zeta function. Here it is important to notice, that k is even.

For our purposes the most important fact is, that it inherits the functional equation of the Eisenstein series. Beside the functional equation for the Eisenstein series we use the well-known functional equation for the ζ-function

$$\zeta(s)=\pi^{s-\frac{1}{2}}\frac{\Gamma(\frac{1-s}{2})}{\Gamma(\frac{s}{2})}\zeta(1-s) \tag{24}$$

and besides (23) we recall the familiar identity for the Gamma-function

$$\Gamma(s)\Gamma(1-s)=\frac{\pi}{\sin(\pi s)}. \tag{25}$$

We start with the functional equation for the Eisenstein series

$$E_{-k}(z,s-\frac{k-1}{2})=\frac{i^k 2^{k+1-2s}\pi\Gamma(2s-k)\zeta(2s-k)}{\zeta(2s-k+1)\Gamma(s+\frac{1}{2})\Gamma(s-k+\frac{1}{2})}E_{-k}(z,\frac{1}{2}+\frac{k}{2}-s)$$

to obtain

$$\tilde{F}(\mathcal{Y},s)=\frac{i^k 2^{k-2s}\pi^{\frac{1}{2}-s}\Gamma(2s-k)\zeta(2s-k)}{\Gamma(s-k+\frac{1}{2})}\mathcal{S}_f(\mathcal{Y},-s+(k+1)/2).$$

Now we apply Eq. (24), which gives

$$\tilde{F}(\mathcal{Y},s)=\frac{i^k 2^{k-2s}\pi^{s-k}\Gamma(2s-k)\Gamma(\frac{k+1}{2}-s)\zeta(1+k-2s)}{\Gamma(s-k+\frac{1}{2})\Gamma(s-\frac{k}{2})}\mathcal{S}_f(\mathcal{Y},-s+(k+1)/2).$$

Next apply (23)

$$\tilde{F}(\mathcal{Y},s)=\frac{i^k 2^{-1}\pi^{-\frac{1}{2}+s-k}\Gamma(\frac{1}{2}-\frac{k}{2}+s)\Gamma(\frac{1}{2}+\frac{k}{2}-s)\zeta(1+k-2s)}{\Gamma(s-k+\frac{1}{2})}\mathcal{S}_f(\mathcal{Y},-s+(k+1)/2).$$

Now from Eq. (25) follows

$$\Gamma(\frac{1}{2}-\frac{k}{2}+s)\Gamma(\frac{1}{2}+\frac{k}{2}-s)=\frac{\pi}{\sin(\pi s+(1-k)\frac{\pi}{2})}$$

and

$$\frac{1}{\Gamma(\frac{1}{2}+s-k)}=\frac{1}{\pi}\Gamma(\frac{1}{2}-s+k)\sin(\pi s+(1-2k)\frac{\pi}{2}).$$

Since k is an even integer

$$\frac{\sin(\pi s+(1-2k)\frac{\pi}{2})}{\sin(\pi s+(1-k)\frac{\pi}{2})}=(-1)^{\frac{k}{2}}$$

and we obtain

$$\tilde{F}(\mathcal{Y},s)=2^{-1}\pi^{-\frac{1}{2}+s-k}\Gamma(\frac{1}{2}+k-s)\zeta(1+k-2s)\mathcal{S}_f(\mathcal{Y},-s+(k+1)/2),$$

which by part (i) of the theorem is the same as $\tilde{F}(\mathcal{Y},k-s)$.

This proves (ii).

(iii) By Stirling's formula and the functional equation $s\tilde{F}(\mathcal{Y},s)$ and $(k-s)\tilde{F}(\mathcal{Y},k-s)$ are bounded on $\Re s=c$ for some suitable c. Because of the functional equation in (ii) the same is true for. By the Phragmen-Lindelöf theorem it follows that we have boundedness of $s\tilde{F}(\mathcal{Y},s)$ in the whole strip $k-c<\Re s<c$ and therefore (iii) follows. □

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