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The Error of Present Value Discounting Conventions in the Case of Uniform Intra-Period Cash Flow

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Zusammenfassung / Abstract:

We compare the estimation errors of several present value discounting conventions (end-of-year, mid-year, and the more recently proposed harmonic mean convention) for a uniform distribution of intra-period cash flow – both continuous and discrete. Our results show that the mid-year convention generally performs best for continuous uniform streams of cash flow and that for discrete uniform streams of cash flow, the mid-year and harmonic mean conventions produce smaller errors than the end-of-year convention. We supply conditions for the order of preference in the case of discrete uniform cash flow distributions and provide some additional insights into the errors of the discounting conventions for both continuous and uniform intra-period cash flow distributions.

Schlagworte / Keywords: valuation, mid-year convention, end-of-year bias, discounted cash flow

1. INTRODUCTION

When valuing a company or project, financial analysts often rely on the end-of-year (EOY) or mid-year (MY) convention to simplify the calculation of present value. Under these conventions, the cash flow for a given period (typically one year) is assumed to occur in one lump sum at either the end or the midpoint of the period, respectively (Fleischer, Mason, and Zhou 1998; Lohmann and Oakford 1984). It is well known that these methods can lead to an estimation error, referred to as the end-of-year bias in the case of EOY, when the actual cash flow is made up of cash inflows and outflows that occur at various points in time throughout the period. An accurate calculation of present value would require a (cumbersome) forecast of the time, sign, and magnitude of each individual cash flow event within the period and the application of corresponding intra-period discounting rates.

McMath (1990) and Anderson, Barber, and Thurston (1997) have developed correction constants for removing the end-of-year bias in the present value of streams of intra-period cash flows that represent elementary seasonal patterns, such as uniform, linear, and geometric cash flow streams.[1] In these cases, practitioners who simplify their present value calculations by employing a "one-size-fits-all" standard discounting convention will incur an error that may easily be quantified. Fleischer, Mason, and Zhou (1998) examine this error for a discrete uniform distribution of cash flow and find that the

[1] These can easily be adjusted for use with the mid-year convention, using a simple factor (cf. Lohmann and Oakford 1984) that converts MY present values to EOY present values.

MY convention is mostly preferable to the EOY convention and continuous approximations.

Andor and Dülk (2013) have recently shown how the harmonic mean of beginning-of-period and end-of-period present values (the harmonic mean convention, HM) can be used to minimize the maximum possible error for an unknown pattern of non-negative intra-period cash flow. Using error plots, they compare the relative errors of the EOY, MY, and HM conventions for several defined distributions of intra-period cash flow (PERT, triangular and "semester estimates") and find that the bias caused by the EOY, MY, and HM conventions for these patterns is small (below 5 percent) when discount rates are below approximately 7 percent, and that the MY and HM conventions are superior to the EOY convention for larger discount rates.

Similar to Fleischer, Mason, and Zhou (1998), we examine the relative error of common discounting conventions for a uniform distribution of cash flow, on the assumption that this most basic setting may roughly apply to a number of situations in practice. Our paper extends their work by providing generalized results from a more rigorous formal analysis and by including the more recent HM convention in the comparison. Our work is also similar to that of Andor and Dülk (2013). In contrast to their approach, we perform a more formal analysis and focus on the case of a simple uniform distribution of cash flow – both discrete and continuous.

The paper is organized as follows: Chapter 2 introduces discounting conventions for intra-period cash flow and their relative errors. Chapter 3 examines the relative errors in the case of discrete uniform intra-period distributions of cash flow, while Chapter 4 considers the continuous case. Chapter 5 concludes.

2. DISCOUNTING CONVENTIONS FOR INTRA-PERIOD CASH FLOW AND RELATIVE ERROR

Consider a single period of cash flow, for example one year. The present value of a discrete stream of intra-period cash flow can then be expressed as (see Andor and Dülk 2013):

$$V = \sum_{q=1}^{Q} C_q (1+r)^{-t_q}, \qquad (1)$$

where Q is the total number of cash flows within the period (note that Q need not be finite), q is an index number representing the order of the cash flows in time, C_q is the q^{th} cash flow within the period, r is the effective interest rate for the period and $t_q \epsilon [0,1]$ is the (continuous) time of occurrence of cash flow C_q expressed as a fraction of the total length of the period. To facilitate the computation of present value, the discrete stream of cash flow over time is sometimes approximated by a continuous function.[2] Accordingly, the present value of a continuous rate of intra-period cash flow $c(t)$ over a period can be calculated as (see Anderson, Barber, and Thurston 1997):

$$V = \int_0^1 c(t) e^{-jt} dt, \qquad (2)$$

[2] Anderson, Barber, and Thurston (1997) believe that present values based on continuous representations of cash flow streams are a good approximation of discrete values when the frequency of sub-annual cash flow is weekly or greater (p. 229).

where $j = \ln(1 + r)$ is the equivalent continuous rate of interest.[3]

Under the EOY (MY) convention, all intra-period cash flows (inflows and outflows) are aggregated and assumed to occur in a single amount at the end (midpoint) of the period. With $C = \sum_{q=1}^{Q} C_q$, the present value for a period is calculated as:

$$\hat{V}_{EOY} = C(1 + r)^{-1}, \tag{3}$$

and

$$\hat{V}_{MY} = C(1 + r)^{-1/2} = \hat{V}_{EOY}(1 + r)^{1/2}. \tag{4}$$

The relative error ε is then defined as the percentage deviation of the estimate $\hat{V}$ from the actual present value V:

$$\varepsilon = \frac{\hat{V}}{V} - 1. \tag{5}$$

Andor and Dülk (2013) propose a novel present value convention, which uses the harmonic mean of the beginning-of-period and the end-of-period present values. Under this harmonic mean (HM) convention, the present value for a period is calculated as:

$$\hat{V}_{HM} = \frac{2}{1/C + 1/C(1 + r)^{-1}} = C(1 + r/2)^{-1} = \hat{V}_{EOY}\frac{1 + r}{1 + r/2}. \tag{6}$$

The HM convention minimizes the maximum possible error in present value for an unknown pattern of non-negative intra-period cash flow.[4] Note from Equations (4) and

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[3] The use of continuous or discrete discount rates should not affect the outcome of the present value calculation if we assume that the compounding period can be divided into any number of shorter periods (see also Dülk 2016, p. 80).

[4] Dülk (2016) generalizes the harmonic mean convention (and the corresponding correction factor) for cases where the intra-period cash flow includes both positive and negative amounts. While his approach

(6) that a simple adjustment factor $S(r)$ with $\hat{V} = \hat{V}_{EOY}S(r)$ can be used to convert an EOY present value to another present value convention (see also Lohmann and Oakford 1984; Andor and Dülk 2013).

minimizes the maximum possible relative error of the present value estimate, it requires the analyst to decompose the cash flow forecast of each period into positive and negative amounts.

3. DISCRETE UNIFORM DISTRIBUTION

Consider m payments of an amount C/m evenly spaced throughout a defined period, where C is finite and can be positive or negative and $1 < m < \infty$. The period discount rate is r, where $r > 0$. The effective discount rate for a sub-period with duration $\Delta t = 1/m$ is then $i = (1+r)^{1/m} - 1$. The present value of such a discrete uniform stream of cash flow can be calculated as (McMath 1990):

$$V = \sum_{k=1}^{m} \frac{C}{m}(1+r)^{-k/m} = C(1+r)^{-1}\frac{r}{mi} = \hat{V}_{EOY}\frac{r}{mi}, \tag{7}$$

where the relative error is $\varepsilon_{EOY} = \frac{\hat{V}_{EOY}}{V} - 1 = \frac{mi}{r} - 1$. The corresponding relative errors for the MY and HM conventions are given in Table 1.[5]

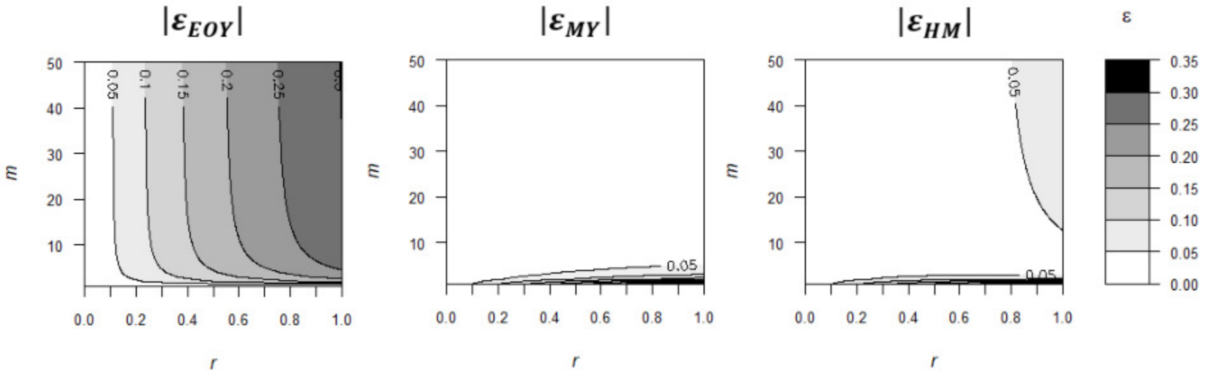

Figure 1. The magnitude of relative errors ε of the end-of-period (EOY), mid-period (MY), and harmonic mean (HM) conventions as a function of the period discount rate r and the number of discrete payments m for a discrete uniform cash flow profile.

[5] Anderson, Barber, and Thurston (1997) provide EOY correction constants for many elementary discrete and continuous seasonal patterns. Using (5), the corresponding relative errors for these patterns can be easily derived.

Figure 1 shows their magnitudes as a function of the number of payments m and the discount rate r. It is easy to see that the relative errors of the MY and HM conventions are mostly smaller than the relative error of the EOY convention. Using the expressions in Table 1, we provide formal proof that the EOY convention is strictly dominated by the HM and MY conventions for any uniform cash flow profile with more than two sub-annual payments and a discount rate greater than zero[6] (see Annex A). In the case of exactly two semi-annual payments ($m = 2$), the EOY convention is still dominated by the HM convention, while the relative errors of the EOY and MY conventions are equivalent.

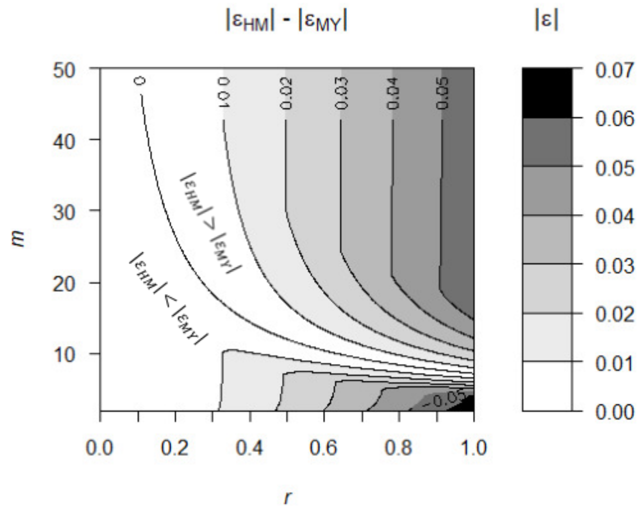

Figure 2. Difference between relative errors ε of the harmonic mean (HM) and mid-period (MY) conventions as a function of the period discount rate r and the number of discrete payments m for a discrete uniform cash flow profile.

The order of preference between the HM convention and the MY convention is less straightforward. We therefore look at the difference in magnitude of the relative errors of these two conventions (shown in Figure 2). It is apparent that for high discount

[6] When the discount rate is zero ($r = 0$), the present value is the simple sum of the intra-period cash flow for each of the three conventions and the corresponding errors are always zero.

rates and high frequencies of intra-period payments, the relative error of the MY convention is smaller than that of the HM convention, and that the opposite relationship holds for small discount rates and small frequencies of cash flow. For example, at a discount rate of $r = 5$ percent (or greater), the relative error of the MY convention is always preferable to the relative error of the HM convention when the number of intra-period payments is 99 or greater (which can be safely assumed for the intra-period cash flow of most firms).

We set the difference between the magnitudes of the HM and MY conventions equal to zero, i.e: $|\varepsilon_{HM}| - |\varepsilon_{MY}| = \left|\frac{1+r}{1+\frac{r}{2}}\frac{mi}{r} - 1\right| - \left|\sqrt{1+r}\frac{mi}{r} - 1\right| = 0$. From this we derive the separating condition (see Annex A):

$$\frac{mi}{r}\left(\sqrt{1+r} + \frac{1+r}{1+\frac{r}{2}}\right) = 2. \tag{8}$$

4. CONTINOUS UNIFORM DISTRIBUTION

At the limit ($m \rightarrow \infty$ and $\Delta t \rightarrow dt$), the discrete intra-period cash flow becomes a continuous uniform stream of cash flow with rate $c(t) = C$. Using (2), the present value expression can be found as

$$\mathrm{V} = \int_0^1 Ce^{-jt}dt = C\frac{(1-e^{-j})}{j}. \tag{9}$$

With (3), the relative error of the end-of-year convention can be shown as

$$\varepsilon_{EOY} = \frac{Ce^{-j}}{C\frac{(1-e^{-j})}{j}} - 1 = \frac{j}{r} - 1. \tag{10}$$

Accordingly, we compute the relative errors for the MY and HM conventions and show them in Table 1 along with a summary of the discrete expressions.

Table 1. Relative errors of cash flow conventions for calculating the present value of discrete and continuous uniform distributions of intra-period cash flow.

	Cash Flow Convention		
	End-of-year	Mid-year	Harmonic Mean
Discrete	$\frac{mi}{r} - 1$	$\sqrt{1+r}\,\frac{mi}{r} - 1$	$\frac{1+r}{1+\frac{r}{2}}\,\frac{mi}{r} - 1$
Continuous	$\frac{j}{r} - 1$	$\sqrt{1+r}\,\frac{j}{r} - 1$	$\frac{1+r}{1+\frac{r}{2}}\,\frac{j}{r} - 1$

Notes: r is the discount rate for the period with $r > -1, r \neq 0$ and $m > 1$ is the number of cash flows per period, i is the intra-period effective discount rate, with $i = (1+r)^{1/m} - 1$, and j is the equivalent continuous discount rate, with $j = ln(1+r)$.

Figure 3 compares the magnitude of the relative errors for these present value conventions as a function of the discount rate and suggests that the relative error is largest for the EOY convention, followed by the HM convention and the MY convention.[7] Using the expressions in Table 1, we provide formal proof that the relation

$$|\varepsilon_{EOY}| > |\varepsilon_{HM}| > |\varepsilon_{MY}|$$

holds for all $r \neq 0$[8] (see Annex B).

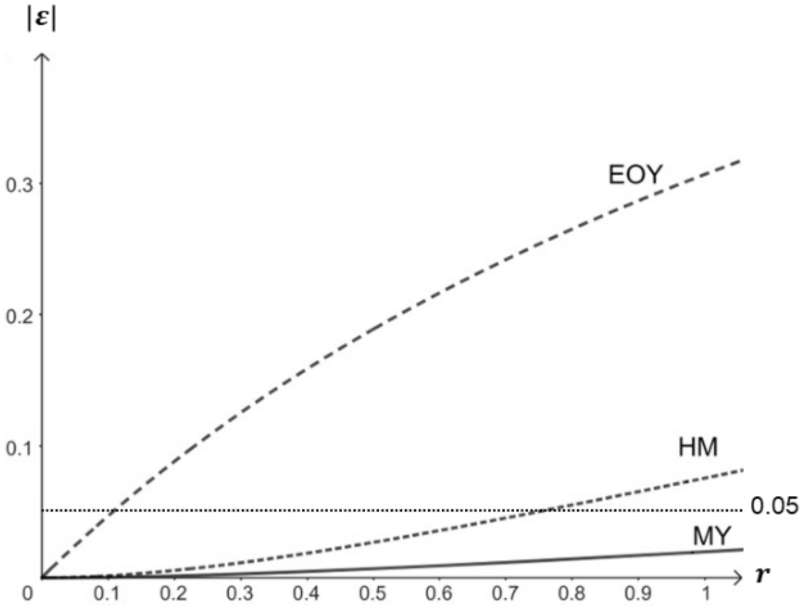

Figure 3. Comparison of the magnitude of relative errors ε of the end-of-period (EOY), mid-period (MY), and harmonic mean (HM) conventions as a function of discount rate r for a continuous uniform intra-period cash flow profile.

Let us consider an error greater than five percent as substantial. The EOY convention exceeds this threshold error for all $r > 10.7$ percent (underscoring McMath's 1990 claim that the EOY bias is not trivial), while the HM and MY conventions exceed this threshold only at far higher discount rates of $r > 74.6$ percent and $r > 205.0$ percent,

[7] Andor and Dülk's (2013) graphical analysis already suggests that the MY convention produces the smallest error. Our uniform continuous cash flow distribution is a special case of their "semester estimates" where the proportion of cash flows is equally distributed ($c = 0.5$).

[8] Again, when the discount rate is zero ($r = 0$), the present value is the simple sum of the intra-period cash flow for each of the three conventions, and the corresponding errors are always zero.

respectively.[9] We can therefore conclude that both the MY and HM conventions deliver reasonably accurate results for most discount rates used in practice, thus largely confirming Andor and Dülk (2013). However, contrary to their claims, the difference between these two conventions is not 'slight'. For discount rates in the range of $0 < r \leq 1$ elementary calculations show that the error of the HM convention is at least 3.8 times larger than that of the MY convention (see Annex B).[10] In this range of discount rates, the maximum error of the MY convention is always less than 2.0 percent.[11]

[9] The magnitude of the error also exceeds 5 percent for negative discount rates: $r < -9.4$ percent (EOY), $r < -42.7$ percent (HM), and $r < -67.2$ percent (MY).

[10] Correspondingly, within this range the error of the EOY convention is at minimum 15.5 times larger than that of the MY convention.

[11] At $r = 1$, $|\varepsilon_{MY}| = \left|\sqrt{2}\ln(2) - 1\right| = 0.0197$.

5. CONCLUSION

Compared to the end-of-year convention and the harmonic mean convention, the mid-year discounting convention yields the smallest error for continuous uniform distributions of intra-period cash flow. This is also true for discrete uniform distributions of intra-period cash flow with moderate to high discount rates and many intra-period cash flow events. For practitioners, our results suggest that, of the three discounting conventions, the mid-year convention will typically yield the smallest error in cases where the firm's cash flow largely resembles a uniform distribution.

6. REFERENCES

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7. ANNEX

A Discrete Case

Proof of $|\boldsymbol{\varepsilon}_{EOY}| \geq |\boldsymbol{\varepsilon}_{MY}|$

In the sequel always assume: $r > 0, m \in \mathbb{N}, m > 1$. Bernoulli's inequality then states that $\left(1+\frac{r}{m}\right)^m > (1+r)$, hence $(1+r)^{1/m} < 1+\frac{r}{m}$. This implies

$$(1+r)^{1/m} - \left(1+\frac{r}{m}\right) = (1+r)^{1/m} - 1 - \frac{r}{m} < 0.$$

Divide by r/m. This yields

$$\varepsilon_{EOY} = \frac{m((1+r)^{1/m} - 1)}{r} - 1 < 0. \tag{A.1}$$

We consider

$$\frac{r}{m}\varepsilon_{EOY} = (1+r)^{1/m} - 1 - \frac{r}{m}$$

and

$$\frac{r}{m}\varepsilon_{MY} = \sqrt{1+r}\left((1+r)^{1/m} - 1\right) - \frac{r}{m}.$$

We substitute $\sqrt{1+r} =: x, 1/m =: a$. Then $x > 1$ and $0 < a \leq 1/2$. $\frac{r}{m}\varepsilon_{EOY}$ and $\frac{r}{m}\varepsilon_{MY}$ can be written as $F(x,a) = x^{2a} - 1 - a(x^2 - 1)$ and $G(x,a) = x(x^{2a} - 1) - a(x^2 - 1)$, respectively.

From (A.1) we deduce that $|F(x,a)| = -F(x,a)$ for all $x > 1$ and $0 < a \leq 1/2$. Depending on the sign of G, the difference $|F(x,a)| - |G(x,a)|$ can be expressed as

$$|F(x,a)| - |G(x,a)| = -F(x,a) + G(x,a) = (x-1)(x^{2a}-1)$$

or

$$|F(x,a)| - |G(x,a)| = -F(x,a) - G(x,a) = 2a(x^2-1) - (x+1)(x^{2a}-1).$$

We show that both expressions are positive for all $x > 1$ and $0 < a < 1/2$. For the first expression this follows immediately, since $x^{2a} > 1$. In the second case, let $H_a(x) \coloneqq 2ax - x^{2a} + 1 - 2a$. Then $H_a(1) = 0$ for all a. $H_a'(x) = 2a - 2ax^{2a-1} = 2a(1 - x^{2a-1})$. Since $2a - 1 < 0$ for $a < 1/2$ and thus $x^{2a-1} < 1$, the derivative is positive and H_a increases strictly monotonously. So H_a is positive for all $x > 1$.

Consequently $|\varepsilon_{EOY}| > |\varepsilon_{MY}|$.

If $a = 1/2$, then $\varepsilon_{MY} = \frac{2}{r}\sqrt{1+r}\left(\sqrt{1+r}-1\right) - 1 = \frac{2}{r}(1+r) - \frac{2}{r}\sqrt{1+r} - 1$. Therefore $\frac{r}{2}\varepsilon_{MY} = \left(1+\frac{r}{2}\right) - \sqrt{1+r} > 0$ (see above) and therefore $|\varepsilon_{MY}| > 0$ for all $r > 0$. So $|\varepsilon_{EOY}| - |\varepsilon_{MY}| = -\varepsilon_{EOY} + \varepsilon_{MY} = 1 - \frac{2}{r}\left(\sqrt{1+r}-1\right) - \frac{2}{r}(1+r) + \frac{2}{r}\sqrt{1+r} + 1 = 0$ for all $r > 0$.

Thus, $|\varepsilon_{EOY}| \geq |\varepsilon_{MY}|$ for all $r > 0, m \in \mathbb{N}, m > 1$. This can be separated into a strict inequality $|\varepsilon_{EOY}| > |\varepsilon_{MY}|$ for all $r > 0$ and $m > 2$, and an equality $|\varepsilon_{EOY}| = |\varepsilon_{MY}|$ for all $r > 0$ and $m = 2$.

Proof of $|\boldsymbol{\varepsilon}_{EOY}| > |\boldsymbol{\varepsilon}_{HM}|$

As before, we put

$$\frac{r}{m}\varepsilon_{EOY} = (1+r)^{1/m} - 1 - \frac{r}{m}$$

and also

$$\frac{r}{m}\varepsilon_{HM} = \frac{1+r}{1+r/2}((1+r)^{1/m} - 1) - \frac{r}{m}.$$

It has already been shown that $\left|\frac{r}{m}\varepsilon_{EOY}\right| = -\frac{r}{m}\varepsilon_{EOY}$ for all $r > 0$ and $m > 1$. Substitute again $\sqrt{1+r} =: x, 1/m =: a$. Then $x > 1$ and $0 < a \leq 1/2$. Thus, $\frac{r}{m}\varepsilon_{EOY}$ and $\frac{r}{m}\varepsilon_{MY}$ can be written as

$$F(x,a) = x^{2a} - 1 - a(x^2 - 1)$$

and

$$E(x,a) = \frac{2x^2}{x^2+1}(x^{2a} - 1) - a(x^2 - 1),$$

respectively.

We have already shown that $F(x,a) < 0$ for all $x > 1$ and $0 < a \leq 1/2$. If $E(x,a) \leq 0$ the difference $|F(x,a)| - |E(x,a)|$ can be written in the form

$$|F(x,a)| - |E(x,a)| = -F(x,a) + E(x,a) = \frac{x^2-1}{x^2+1}(x^{2a} - 1),$$

which is obviously positive for all $x > 1$ and $0 < a \leq 1/2$.

If $E(x,a) > 0$ then

$$|F(x,a)| - |E(x,a)| = -F(x,a) - E(x,a)$$

$$= 2a(x^2 - 1) - \frac{3x^2 + 1}{x^2 + 1}(x^{2a} - 1),$$

$$= (x^2 + 1)^{-1}\big(2a(x^4 - 1) - (3x^2 + 1)(x^{2a} - 1)\big).$$

This is clearly positive for $a = 1/2$, $x > 1$ and so we look more closely at the case $0 < a < 1/2$.

First we claim that

$$(x - 1) > \frac{x^{2a} - 1}{2a} \tag{A.2}$$

for $2a < 1$ and all $x > 1$. Indeed, this follows because for $x = 1$ both sides coincide for any a, and for $x > 1$ we compare their derivatives: $\frac{d}{dx}(x - 1) = 1 > x^{2a-1} = \frac{d}{dx}(x^{2a} - 1)/2a$.

This proves our claim.

Second, notice that $x^3 + x^2 + x + 1 > 3x^2 + 1$. This follows because $x^3 + x^2 + x + 1 - 3x^2 - 1 = x^3 - 2x^2 + x = x(x - 1)^2 > 0$ for $x > 1$.

Now we can conclude

$$2a(x^4 - 1) = 2a(x - 1)(x^3 + x^2 + x + 1) > (x^{2a} - 1)(x^3 + x^2 + x + 1)$$
$$> (3x^2 + 1)(x^{2a} - 1)$$

and hence $-F(x,a) - E(x,a) > 0$ as desired.

Derivation of the Condition for Equality $|\varepsilon_{HM}| = |\varepsilon_{MY}|$

$$|\varepsilon_{HM}| - |\varepsilon_{MY}| = \left|\frac{m(1+r)}{r(1+r/2)}((1+r)^{\frac{1}{m}}-1)-1\right| - \left|\frac{m\sqrt{1+r}}{r}((1+r)^{\frac{1}{m}}-1)-1\right| = 0$$

$1 + r/2 > \sqrt{1+r}$ for all $r > -1, r \neq 0$, since $(1+r/2)^2 = 1 + r + r^2/4 > 1 + r$.

Therefore, $\sqrt{1+r} = \frac{1+r}{\sqrt{1+r}} > \frac{1+r}{1+\frac{r}{2}}$ for these values. Thus,

$$\varepsilon_{HM} \coloneqq \frac{m(1+r)}{r\left(1+\frac{r}{2}\right)}((1+r)^{\frac{1}{m}}-1)-1 > \frac{m\sqrt{1+r}}{r}((1+r)^{\frac{1}{m}}-1) \ -1 \eqqcolon \varepsilon_{MY}$$

for all $r > -1, r \neq 0$. $|\varepsilon_{HM}| - |\varepsilon_{MY}| = 0$ only occurs if $\varepsilon_{HM} = -\varepsilon_{MY}$ and therefore,

$$\frac{m((1+r)^{\frac{1}{m}}-1)}{r}\left(\sqrt{1+r}+\frac{1+r}{1+\frac{r}{2}}\right) = 2.$$

B Continuous Case

Proof of $|\boldsymbol{\varepsilon_{EOY}}| > |\boldsymbol{\varepsilon_{HM}}| > |\boldsymbol{\varepsilon_{MY}}|$

Let $j \in \mathbb{R}, j \neq 0$. With $r = e^j - 1$ we derive the following expressions from Table 1: $\varepsilon_{EOY} = \frac{j}{e^j-1} - 1$, $\varepsilon_{MY} = e^{j/2}\frac{j}{e^j-1} - 1$ and $\varepsilon_{HM} = \frac{2}{1+e^{-j}}\frac{j}{e^j-1} - 1$.

(i) The relative error of the harmonic mean convention can be expressed as $\varepsilon_{HM} = \frac{2}{1+e^{-j}}\frac{j}{e^j-1} - 1 = \frac{2j}{e^j-e^{-j}} - 1$. The real exponential function e^j can be defined by the power series $e^j = 1 + j + \frac{j^2}{2!} + \frac{j^3}{3!} + \cdots$ and accordingly $e^{-j} = 1 - j + \frac{j^2}{2!} - \frac{j^3}{3!} \pm \cdots$. Therefore, $(e^j - e^{-j}) = 2j + 2\frac{j^3}{3!} + 2\frac{j^5}{5!} + \cdots$ and $(e^j - e^{-j})/2j = 1 + \frac{j^2}{3!} + \frac{j^4}{5!} + \cdots$. With $j \neq 0$ follows $\frac{2j}{e^j-e^{-j}} < 1$ and $|\varepsilon_{HM}| = -\varepsilon_{HM}$.

(ii) The relative error of the mid-year convention can be expressed as $\varepsilon_{MY} = e^{j/2}\frac{j}{e^j-1} - 1 = \frac{j}{e^{j/2}-e^{-j/2}} - 1$. Thus, evidently, $\varepsilon_{MY}(j) = \varepsilon_{HM}(j/2)$. In (i) we have shown that $\frac{2j}{e^j-e^{-j}} < 1$ for all $j \neq 0$. As $j/2 \neq 0 \Leftrightarrow j \neq 0$ it follows that $\frac{j}{e^{j/2}-e^{-j/2}} < 1$ and $|\varepsilon_{MY}| = -\varepsilon_{MY}$ for all $j \neq 0$.

(iii) We can easily see that $(e^j - e^{-j})/2j = 1 + \frac{j^2}{3!} + \frac{j^4}{5!} + \cdots$ is greater than 1 and strictly monotonically increasing (decreasing) for all $j > 0$ ($j < 0$). Thus, $\frac{2j}{e^j-e^{-j}}$ is less than 1 and strictly monotonically decreasing (increasing) and accordingly, $|\varepsilon_{HM}| = -\varepsilon_{HM} = 1 - \frac{2j}{e^j-e^{-j}}$ is strictly increasing (decreasing). Therefore, $|\varepsilon_{MY}(j)| = |\varepsilon_{HM}(j/2)| < |\varepsilon_{HM}(j)|$ for all $j \neq 0$.

(iv) The magnitude of the relative error of the end-of-year convention can be expressed as $|\varepsilon_{EOY}| = \left|\frac{j}{e^j-1} - 1\right|$. Using the power series once again, $e^j - 1 = j + \frac{j^2}{2!} + \frac{j^3}{3!} + \cdots$ and $\frac{e^j-1}{j} = 1 + \frac{j}{2!} + \frac{j^2}{3!} + \cdots > 1$ for all $j > 0$. Thus, $\frac{j}{e^j-1} < 1$and $|\varepsilon_{EOY}| = -\varepsilon_{EOY}$ for all $j > 0$. In the case of $j < 0$ we examine $f(j) = \frac{j}{e^j-1} = \frac{|j|}{1-e^{-|j|}} = \frac{ze^z}{e^z-1}$, where $z \coloneqq |j|$. Consider the Taylor series $ze^z = z + z^2 + \frac{z^3}{2} + \frac{z^4}{3!} + \cdots$ and $e^z - 1 = z + \frac{z^2}{2} + \frac{z^3}{3!} + \frac{z^4}{4!} + \cdots$. It is immediately apparent that $ze^z > e^z - 1$ for $z > 0$. Therefore $f(j) > 1$ and thus $|\varepsilon_{EOY}| = \varepsilon_{EOY}$ for all $j < 0$.

(v) To identify the direction of $|\varepsilon_{EOY}| - |\varepsilon_{HM}|$, we analyze $|\varepsilon_{EOY}| - |\varepsilon_{HM}| = -\varepsilon_{EOY} + \varepsilon_{HM} = -\left(\frac{j}{e^j-1} - 1\right) + \left(\frac{2j}{e^j-e^{-j}} - 1\right) = \frac{2j}{e^j-e^{-j}} - \frac{j}{e^j-1}$ for all $j > 0$. As $\frac{e^j-1}{j} = 1 + \frac{j}{2!} + \frac{j^2}{3!} + \cdots > 1 + \frac{j^2}{3!} + \frac{j^4}{5!} + \cdots = \frac{e^j-e^{-j}}{2j}$for all $j > 0$, therefore $\frac{j}{e^j-1} < \frac{2j}{e^j-e^{-j}}$ and thus $|\varepsilon_{EOY}| > |\varepsilon_{HM}|$ for all $j > 0$. Correspondingly, we analyze $|\varepsilon_{EOY}| - |\varepsilon_{HM}| = \varepsilon_{EOY} + \varepsilon_{HM} = \left(\frac{j}{e^j-1} - 1\right) + \left(\frac{2j}{e^j-e^{-j}} - 1\right)$ for all $j < 0$. We know from (iv) that $\frac{j}{e^j-1} - 1 > 0$ and from (i) that $\frac{2j}{e^j-e^{-j}} - 1 < 0$. We also know that $\frac{e^j-1}{j} = 1 + \frac{j}{2!} + \frac{j^2}{3!} + \cdots < 1 + \frac{j^2}{3!} + \frac{j^4}{5!} + \cdots = \frac{e^j-e^{-j}}{2j}$for all $j < 0$, and therefore $\frac{j}{e^j-1} - 1 > \frac{2j}{e^j-e^{-j}} - 1$. Thus $\varepsilon_{EOY} + \varepsilon_{HM} > 0$ and $|\varepsilon_{EOY}| > |\varepsilon_{HM}|$ for all $j < 0$.

(vi) It follows from (iii) and (v) that $|\varepsilon_{EOY}| > |\varepsilon_{HM}| > |\varepsilon_{MY}|$ for all $j \neq 0$. ∎

Relative Error Ratios

We plot the relative errors of the EOY and the HM conventions against that of the MY convention. For $r > 0$, the relative errors of all three conventions are less than zero and we do not need to consider the sign of the ratios. Therefore

$$\frac{|\varepsilon_{EOY}|}{|\varepsilon_{MY}|} = \frac{\varepsilon_{EOY}}{\varepsilon_{MY}} = \frac{\ln(1+r) - \mathrm{r}}{\sqrt{1+r}\,\ln(1+r) - r}$$

and

$$\frac{|\varepsilon_{HM}|}{|\varepsilon_{MY}|} = \frac{\varepsilon_{HM}}{\varepsilon_{MY}} = \frac{\dfrac{1+r}{1+\dfrac{r}{2}}\ln(1+r) - \mathrm{r}}{\sqrt{1+r}\ \ln(1+r) - r}.$$

The error ratios are largest at discount rates close to zero, with $\lim_{j\to 0}\frac{\varepsilon_{HM}(j)}{\varepsilon_{MY}(j)} = 4$ and $\lim_{j\to 0}\frac{\varepsilon_{EOY}(j)}{\varepsilon_{MY}(j)} = \infty$ (see below). In the range of $0 < r \leq 1$, the error ratios decrease monotonically with r (see Figure B.1). The lower bounds in this range are at $r = 1$, with $\frac{\varepsilon_{EOY}}{\varepsilon_{MY}} = \frac{ln2-1}{\sqrt{2}ln2-1} \approx 15.54$ and $\frac{\varepsilon_{HM}}{\varepsilon_{MY}} = \frac{4ln2-3}{3\sqrt{2}ln2-3} \approx 3.84$.

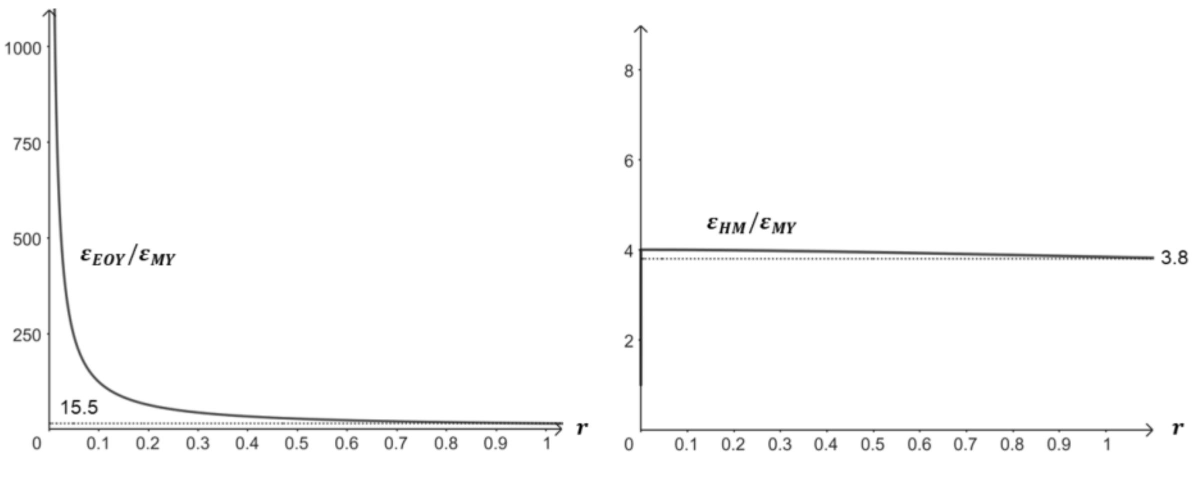

Figure B.1. Ratios of relative errors ε of the end-of-period (EOY), mid-period (MY), and harmonic mean (HM) conventions as a function of discount rate r for a continuous uniform intra-period cash flow profile.

Limits of Relative Error Ratios at $j = 0$

Observe that $\varepsilon_{HM} = \frac{2}{1+e^{-j}} \cdot \frac{j}{e^{j}-1} - 1 = \frac{2j}{e^{j}-e^{-j}} - 1 = \frac{j}{\sinh(j)} - 1 = \frac{j-\sinh(j)}{\sinh(j)}$. Since $\varepsilon_{MY}(j) = \varepsilon_{HM}(j/2)$ we obtain

$$\frac{\varepsilon_{HM}(j)}{\varepsilon_{MY}(j)} = \frac{j - \sinh(j)}{j/2 - \sinh(j/2)} \cdot \frac{\sinh(j/2)}{\sinh(j)}.$$

We want to study the behavior of the quotient near $j = 0$. Remark that $\sinh(j)' = \cosh(j)$, $\cosh'(j) = \sinh(j)$, $sinh(0) = 0$ and $cosh(0) = 1$. Then we obtain by a repeated use of the rule of l'Hospital

$$\lim_{j\to 0} \frac{j - \sinh(j)}{\frac{j}{2} - \sinh\left(\frac{j}{2}\right)} = \lim_{j\to 0} \frac{1 - \cosh(j)}{\frac{1}{2} - \frac{1}{2} \cdot \cosh\left(\frac{j}{2}\right)} = \lim_{j\to 0} \frac{\sinh(j)}{1/4 \cdot \sinh(j/2)} = \lim_{j\to 0} \frac{\cosh(j)}{1/8 \cdot \cosh(j/2)} = 8.$$

Similarly, $\lim_{j\to 0} \frac{\sinh(j/2)}{\sinh(j)} = \frac{1}{2}$ and therefore

$$\lim_{j\to 0} \frac{\varepsilon_{HM}(j)}{\varepsilon_{MY}(j)} = 4.$$

Next, we look at $\frac{\varepsilon_{MY}(j)}{\varepsilon_{EOY}(j)} = \frac{\frac{j}{2} - \sinh\left(\frac{j}{2}\right)}{\sinh\left(\frac{j}{2}\right)} \cdot \frac{e^{j}-1}{j+1-e^{j}}$. (Here we use $\varepsilon_{EOY}(j) = \frac{j}{e^{j}-1} - 1$ $= \frac{j+1-e^{j}}{e^{j}-1}$.). Similar reasoning as before gives

$$\lim_{j\to 0} \frac{j/2 - \sinh(j/2)}{j + 1 - e^{j}} = \lim_{j\to 0} \frac{1/2 - 1/2 \cdot \cosh(j/2)}{1 - e^{j}} = \lim_{j\to 0} \frac{1/4 \cdot \sinh(j/2)}{-e^{j}} = 0$$

and

$$\lim_{j\to 0} \frac{e^{j} - 1}{\sinh(j/2)} = \lim_{j\to 0} \frac{e^{j}}{1/2 \cdot \cosh(j/2)} = 2.$$

Therefore $\lim_{j \to 0} \frac{\varepsilon_{MY}(j)}{\varepsilon_{EOY}(j)} = 0$ and

$$\lim_{j \to 0} \frac{\varepsilon_{EOY}(j)}{\varepsilon_{MY}(j)} = \infty.$$

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