Leibniz FH

Research Paper Series
Nr. 12 | Dezember 2023

Planning a Joint Venture
Decision Support by Linear Vector Duality Theory

Autoren:
Wilhelm Rödder, Andreas Dellnitz

Leibniz-Fachhochschule

Autoren:

Andreas Dellnitz
Leibniz-Fachhochschule
dellnitz@leibniz-fh.de

Wilhelm Rödder
Leibniz-Fachhochschule

ISSN 2511-7491

Redaktion: Robin Christmann

Leibniz FH
University of Applied Sciences
Research Paper No. 12

Planning a Joint Venture: Decision Support by Linear Vector Duality Theory

WILHELM RÖDDER

ANDREAS DELLNITZ

Zusammenfassung / Abstract:

Whereas duality for linear vector optimum problems often is just a mere mathematical construct (l'art pour l'art), we define a family of dual problems which help decision makers in a concrete economic situation. Entrepreneurs want to join activities for producing different products with similar technologies in a new factory. Each of them provides resources relative to the others' consideration of his/her economic objective. They all agree to a process of aggregating their different interests by a new duality concept. This paper first presents the duality concept and shows under which conditions a satisfying aggregation of the entrepreneurs' different objectives is possible. Numerical examples help to make this aggregation transparent. An application then demonstrates the capability of the new concept.

Schlagworte / Keywords: linear vector optimum problems; duality theory; joint venture planning

1 Introduction

The field of vector or multi-objective optimization has experienced a boom in research interest in recent years, as Fig. 1 illustrates.. In total, a search on the Web of Science platform yields over 5,000 peer-reviewed articles starting in 1980, using the three terms “vector optimization”, “multi-objective optimization”, and “multiple objective optimization”. These contributions open up a wide range of applications such as production planning, financial planning, aerospace engineering, educational planning, infrastructure management, ecological modeling etc.; see, e.g., Yenisey and Yagmahan (2014), Gardi et al. (2016), Williams and Kendall (2017), Kleine and Dellnitz (2017), Mansour et al. (2019), Zajac and Huber (2021). This emerging trend towards vector optimization is due to the fact that our modern society no longer wants decision problems to be reduced to a one-dimensional view, but rather the simultaneous consideration of multiple facets of an issue is the new credo.

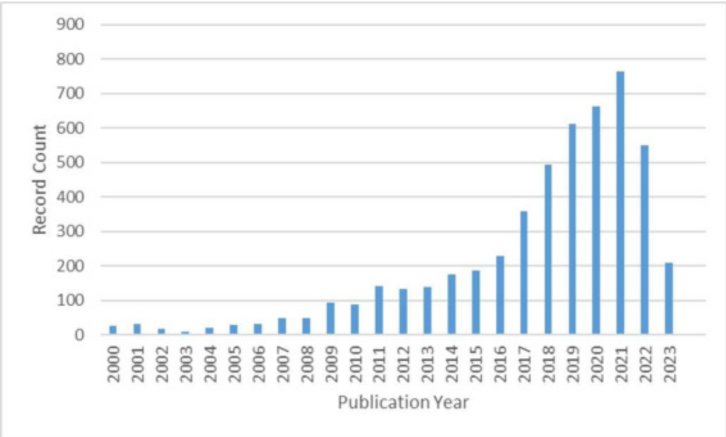


Figure 1: Number of publications per year

It is surprising, however, that the term duality (theory) can be found in only 115 of this huge set of manuscripts. And more importantly, none of these manuscripts uses duality theory to rethink data structures in the context of joint venture planning, which is what this paper is about.

Duality for classical linear programming was one of the highlights in the field of Operation Research (OR) already in the 50s of the last century. The primal problem joins economic activities $\mathbf{x}$ and a technology matrix $\mathbf{A}$, a resources vector $\mathbf{b}$, and vector $\mathbf{c}$ of the activities' yield or profit. The dual problem then allows the calculation of the resources' (dual) prices. All this we find in early textbooks like Charnes et al. (1953), Hadley (1962), Collatz and Wetterling (1966), already. Very soon, the OR community criticized the single objective concept and started considering Linear Vector Optimization Problems (LVOPs); cf. von Neumann and Morgenstern (1944), Ijiri (1965), Mangasarian (1969). Now, it needed a new comprehension of optimality for the vector-valued objective. And consequently, it needed a new duality concept or – better – duality concepts. We name early works of Schönfeld (1964), Kuhn and Tucker (1951), Kornbluth (1974), Isermann (1974), Rödder and Leberling (1976). The latter is a working paper and, thus, it is not accessible to a broader audience. A short, but incomplete introduction the reader also finds in Rödder (1980). To make this paper self-contained, we repeat some basic ideas from Rödder and Leberling (1976) and Rödder (1980). Recent theoretical work on duality theories and corresponding embeddings or connections between them can be found, for example, in Azimov (2008), Hamel (2011), Heyde and Löhne (2011), Heyde and Schrage (2013).

This contribution is organized as follows: Section 2 gives preliminaries. In Section 3.1, we define a family of dual LVOPs and study their properties. Section 3.2 is central in as much it treats the question of a satisfying aggregation of the LVOPs' objectives. Illustrative numerical examples are put into Appendix 1. Section 4 prepares the economical point of view and gives dimensions of all parameters and variables in the dual models. Section 5 shows the capability of the new theory for application in the context of joint ventures. Section 6 is a summary and sketches possible future work.

2 Preliminaries

This section is a must to give all symbols and relations a clear meaning in the remainder of this contribution. For $\mathbf{x}^\mathsf{T} = (x_1, \dots, x_r) \in \mathbb{R}^r$, $\mathbf{y}^\mathsf{T} = (y_1, \dots, y_r) \in \mathbb{R}^r$ we have

$$\begin{aligned} \mathbf{x} \geqq \mathbf{y} &\quad \text{iff } x_i \geqq y_i \ \forall i \in \{1, \dots, r\}, \\ \mathbf{x} \geq \mathbf{y} &\quad \text{iff } \mathbf{x} \geqq \mathbf{y} \text{ and } \mathbf{x} \neq \mathbf{y}, \\ \mathbf{x} > \mathbf{y} &\quad \text{iff } x_i > y_i \ \forall i \in \{1, \dots, r\}, \\ \mathbf{x} \leqq (\leq, <) \mathbf{y} &\quad \text{iff } -\mathbf{x} \geqq (\geq, >) -\mathbf{y}. \end{aligned}$$

If $\geqq (\geq, >)$ is not true, we write $\ngeqq (\ngeq, \ngtr)$, for short. For $\mathbf{Z} \subseteq \mathbb{R}^m$, we call $\mathbf{z}^\circ \in \mathbf{Z}$ efficient in $\mathbf{Z}$ iff $\forall \mathbf{z} \in \mathbf{Z}$ $(\mathbf{z} \geq \mathbf{z}^\circ \Rightarrow \mathbf{z} = \mathbf{z}^\circ)$.

If F is a mapping $\mathbb{R}^n \supseteq \mathbf{X} \xrightarrow{F} \mathbb{R}^m$ and $\mathbf{Z} = F(\mathbf{X})$, then $\mathbf{x}^\circ \in \mathbf{X}$ is called functional efficient (f.e.) for F in $\mathbf{X}$ iff $F(\mathbf{x}^\circ)$ is efficient in $\mathbf{Z}$. $\text{'}\underset{\mathbf{x}\in\mathbf{X}}{\text{Max}}\text{'}\ F(\mathbf{x})$ is the task to find all f.e. solutions for F in X, $\text{'}\underset{\mathbf{x}\in\mathbf{X}}{\text{Min}}\text{'}\ F(\mathbf{x})$ is equivalent to $-\text{'}\underset{\mathbf{x}\in\mathbf{X}}{\text{Max}}\text{'}\ -F(\mathbf{x})$. For the linear vector maximum problem, we use classical indexation and write

$$\text{'Max' } \mathbf{C}^\mathsf{T}\mathbf{x} \quad \text{s.t. } \mathbf{A}\mathbf{x} \leqq \mathbf{b}, \mathbf{x} \geqq \mathbf{0} \tag{1}$$

($\mathbf{C}^\mathsf{T}(K \times n)$, $\mathbf{A}(m \times n)$, $\mathbf{x}$, $\mathbf{0} \in \mathbb{R}^n$, $\mathbf{b} \in \mathbb{R}^m$).

Lemma 1. $\mathbf{x}^\circ$ *is a f.e. solution iff there exists an* $\mathbf{y} \in \mathbb{R}^K, \mathbf{y} > \mathbf{0}$ *such that* $\mathbf{x}^\circ$ *solves*

$$\text{Max } \mathbf{y}^\mathsf{T}\mathbf{C}^\mathsf{T}\mathbf{x} \quad \text{s.t. } \mathbf{A}\mathbf{x} \leqq \mathbf{b}, \mathbf{x} \geqq \mathbf{0}. \tag{2}$$

For a proof of Lemma 1, see Gale et al. (1951) and Isermann (1974), Theorem 2.11. Finding all f.e. solutions for (1) is equivalent to finding all solutions of the parametric problem (2). We mention the obvious fact that Lemma 1 also holds with the additional restriction $\sum_{k=1}^{K} y_k = 1$.

3 A duality theory for LVOPs

3.1 The theory and basic results

Now, we are ready to study a special duality concept for LVOPs. In so doing, we define the family of pairs of equations

$$\text{'Max' } \mathbf{C}^\mathsf{T}\mathbf{x} \quad \text{s.t. } \mathbf{A}\mathbf{x} \leqq \mathbf{B}\mathbf{y}, \mathbf{x} \geqq \mathbf{0} \tag{3a}$$

$$\text{'Min' } \mathbf{u}^\mathsf{T}\mathbf{B} \quad \text{s.t. } \mathbf{u}^\mathsf{T}\mathbf{A} \geqq \mathbf{y}^\mathsf{T}\mathbf{C}^\mathsf{T}, \mathbf{u} \geqq \mathbf{0} \tag{3b}$$

($\mathbf{x} \in \mathbb{R}^n$, $\mathbf{u} \in \mathbb{R}^m$, $\mathbf{y} \in \mathbb{R}^K$, $\mathbf{C}^\mathsf{T}(K \times n)$, $\mathbf{A}(m \times n)$, $\mathbf{B}(m \times K)$. Here and in the remainder of this paper, $\mathbf{y}$ is a *common* parameter for (3a) and (3b), with $\mathbf{y} > \mathbf{0}$, $\sum_{k=1}^{K} y_k = 1$.

Definition 1. *For a fix* $\mathbf{y} = \bar{\mathbf{y}}$*, Eqs.* (3a) *and* (3b) *are a pair of dual LVOPs, with* (3a) *being the primal and* (3b) *the dual one.*

$\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}^\mathsf{T}$ are fix matrices, $\mathbf{y}$ is a parameter and $\mathbf{x}$, $\mathbf{u}$ are variables. The following Lemma 2 is a first justification for naming Eqs. (3a) and (3b) dual LVOPs.

Lemma 2.

i) If both Eqs. (3a) *and* (3b) *are feasible for* $\mathbf{y} = \bar{\mathbf{y}}$*, then* $\mathbf{C}^\mathsf{T}\mathbf{x} \ngeq (\mathbf{u}^\mathsf{T}\mathbf{B})^\mathsf{T}\ \forall \mathbf{x}, \mathbf{u}$.

ii) If $\mathbf{C}^\mathsf{T}\bar{\mathbf{x}} = (\bar{\mathbf{u}}^\mathsf{T}\mathbf{B})^\mathsf{T}$ *for some feasible* $\bar{\mathbf{x}}, \bar{\mathbf{u}}$*. then* $\bar{\mathbf{x}}$ *is f.e. for* (3a) *and* $\bar{\mathbf{u}}$ *is f.e. for* (3b).

Proof.

i) Assuming $\mathbf{C}^\mathsf{T}\mathbf{x} \geq (\mathbf{u}^\mathsf{T}\mathbf{B})^\mathsf{T}$ for some $\mathbf{x}, \mathbf{u}$ implies $\bar{\mathbf{y}}^\mathsf{T}\mathbf{C}^\mathsf{T}\mathbf{x} > \mathbf{u}^\mathsf{T}\mathbf{B}\bar{\mathbf{y}}$. On the other hand

$$\text{Max } \bar{\mathbf{y}}^\mathsf{T}\mathbf{C}^\mathsf{T}\mathbf{x} \quad \text{s.t. } \mathbf{A}\mathbf{x} \leqq \mathbf{B}\bar{\mathbf{y}}, \mathbf{x} \geqq \mathbf{0}$$

and

$$\text{Min } \mathbf{u}^\mathsf{T}\mathbf{B}\bar{\mathbf{y}} \quad \text{s.t. } \mathbf{u}^\mathsf{T}\mathbf{A} \geqq \bar{\mathbf{y}}^\mathsf{T}\mathbf{C}^\mathsf{T}, \mathbf{u} \geqq \mathbf{0}$$

are classical feasible linear problems (LPs) with $\bar{\mathbf{y}}^\mathsf{T}\mathbf{C}^\mathsf{T}\mathbf{x} \leqq \mathbf{u}^\mathsf{T}\mathbf{A}\mathbf{x} \leqq \mathbf{u}^\mathsf{T}\mathbf{B}\bar{\mathbf{y}}$, contradicting the assumption.

ii) Assuming $\mathbf{C}^\mathsf{T}\bar{\mathbf{x}}$ not f.e. for (3a) implies the existence of an $\bar{\bar{\mathbf{x}}}$ and $\mathbf{C}^\mathsf{T}\bar{\bar{\mathbf{x}}} \geq \mathbf{C}^\mathsf{T}\bar{\mathbf{x}} = (\bar{\mathbf{u}}^\mathsf{T}\mathbf{B})^\mathsf{T}$, contradicting *i)*.
Assuming $\bar{\mathbf{u}}^\mathsf{T}\mathbf{B}$ not f.e. for (3b) implies the same contradiction.

$\square$

Lemma 2 impedes the strict dominance of the primal over the dual function value. Furthermore, equality of such values yields functional efficiency for both. That is what duality should do. But there is still an open question whether equality $\mathbf{C}^\top\mathbf{x} = (\mathbf{u}^\top\mathbf{B})^\top$ always is possible. It is not always possible but yet under weak conditions. More on that in the following section.

3.2 Duality and equality $\mathbf{C}^\top\mathbf{x} = (\mathbf{u}^\top\mathbf{B})^\top$

Motzkin in 1936 published his PhD thesis "Beiträge zur Theorie der linearen Ungleichungen" (Motzkin, 1936). For a presentation easier accessible, cf. Mangasarian (1969), p. 28. We put Motzkin's result as a theorem, using the notation as in our LVOP context.

Theorem 1. *Eq.* (4)

$$\begin{array}{rcl} 0\mathbf{x} + 0\mathbf{u} + \mathbf{I}\mathbf{y} & > & \mathbf{0} \\ -\mathbf{A}\mathbf{x} + 0\mathbf{u} + \mathbf{B}\mathbf{y} & \geqq & \mathbf{0} \\ \mathbf{I}\mathbf{x} + 0\mathbf{u} + 0\mathbf{y} & \geqq & \mathbf{0} \\ 0\mathbf{x} + \mathbf{A}^\top\mathbf{u} - \mathbf{C}\mathbf{y} & \geqq & \mathbf{0} \\ 0\mathbf{x} + \mathbf{I}\mathbf{u} + 0\mathbf{y} & \geqq & \mathbf{0} \\ \mathbf{C}^\top\mathbf{x} - \mathbf{B}^\top\mathbf{u} + 0\mathbf{y} & = & \mathbf{0} \end{array} \tag{4}$$

or Eq. (5)

$$\begin{array}{rcl} 0\mathbf{z}^1 + (-\mathbf{A}^\top, \mathbf{I}, \mathbf{0}, \mathbf{0})\mathbf{z}^2 + \mathbf{C}\mathbf{z}^3 & = & \mathbf{0} \\ 0\mathbf{z}^1 + (\mathbf{0}, \mathbf{0}, \mathbf{A}, \mathbf{I})\mathbf{z}^2 - \mathbf{B}\mathbf{z}^3 & = & \mathbf{0} \\ \mathbf{I}\mathbf{z}^1 + (\mathbf{B}^\top, \mathbf{0}, -\mathbf{C}^\top, \mathbf{0})\mathbf{z}^2 + 0\mathbf{z}^3 & = & \mathbf{0} \\ \mathbf{z}^1 \geq \mathbf{0},\ \mathbf{z}^2 \geqq \mathbf{0},\ \mathbf{z}^3\, arbitrary & & \end{array} \tag{5}$$

has a solution, but never both.

We observe that Eq. (4) is the equation Eq. (3a), (3b) including $\mathbf{C}^\top\mathbf{x} = (\mathbf{u}^\top\mathbf{B})^\top$. After renaming variables and adequate transpositions, Eq. (5) reads

$$\mathbf{u}^\top\mathbf{A} \geqq \mathbf{y}^\top\mathbf{C}^\top,\ \mathbf{A}\mathbf{x} \leqq \mathbf{B}\mathbf{y},\ \mathbf{C}^\top\mathbf{x} \geq (\mathbf{u}^\top\mathbf{B})^\top,\ \mathbf{x} \geqq \mathbf{0},\ \mathbf{u} \geqq \mathbf{0} \text{ and } \mathbf{y} \text{ arbitrary!} \tag{5'}$$

The solution of Eq. (4) meets mathematical rationale: There exist a positive $\bar{\mathbf{y}} > \mathbf{0}$ and a feasible $\bar{\mathbf{x}}$, $\bar{\mathbf{u}}$ for Eqs. (3a) and (3b) with equal primal and dual objective function values. If Eq. (4) has no solution, then Eq. (5) does. Eq. (5) is against mathematical rationale, as the following lemma illustrates.

Lemma 3. *Each solution of Eq.* (5′) *necessarily implies in* $y_k \leqq 0$ *for some* k*.*

Proof.
From Lemma 2 *i)* we know that for feasible solutions $\mathbf{x}$ and $\mathbf{u}$ in the dual pair (3a) and (3b) and $\mathbf{y} = \bar{\mathbf{y}}$ always $\mathbf{C}^\mathsf{T}\mathbf{x} \ngeq (\mathbf{u}^\mathsf{T}\mathbf{B})^\mathsf{T}$ holds. So in Eq. (5′) $y_k \leqq 0$ for some k. □

For $y_k = 0$, the k-th objective function does not exist in the primal LVOP. $y_k < 0$, for some k, is a severe aberration from our duality concept. The consequences of such aberration will be elaborated in the application section. We mention again the fact that all observations in this section remain true with the additional constraint $\sum_{k=1}^{K} y_k = 1$.

Please note: The non-existence of a solution of Eq. (4) is not a flaw of the duality concept but rather a flaw of the choice of the data $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}^\mathsf{T}$. We invite the reader to scroll to Appendix A1. There we show little numerical examples justifying the following statements: There exist data $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}^\mathsf{T}$ for which both

- (3a) and (3b) are infeasible;
- (3a) and (3b) are feasible and $\mathbf{C}^\mathsf{T}\mathbf{x} = (\mathbf{u}^\mathsf{T}\mathbf{B})^\mathsf{T}$ for all f.e. $\mathbf{x}$, $\mathbf{u}$;
- (3a) and (3b) are feasible and $\mathbf{C}^\mathsf{T}\mathbf{x} \neq (\mathbf{u}^\mathsf{T}\mathbf{B})^\mathsf{T}$ for all f.e. $\mathbf{x}$, $\mathbf{u}$ and hence Eq. (5′) holds;
- (3a) and (3b) are feasible and $\mathbf{C}^\mathsf{T}\mathbf{x} = (\mathbf{u}^\mathsf{T}\mathbf{B})^\mathsf{T}$ for some f.e. $\mathbf{x}$, $\mathbf{u}$ and $\mathbf{C}^\mathsf{T}\mathbf{x} \neq (\mathbf{u}^\mathsf{T}\mathbf{B})^\mathsf{T}$ for other f.e. $\mathbf{x}$, $\mathbf{u}$.

The right choice of $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}^\mathsf{T}$ is a challenging task for a decision maker, using the presented duality concept. The next section paves the way from a mathematical to an economical concept.

4 From mathematical towards economical LVOPs

As announced in the introduction, entrepreneurs or stakeholders want to join production activities in a new factory. They plan common activities $\mathbf{x}^\mathsf{T} = (x_1, \ldots, x_j, \ldots, x_n)$, with $dim(x_j)$ being the dimension of x_j of product P_j. $\mathbf{A} = (a_{ij})_{m,n}$ is the technology matrix and m the number of resources, with re_i, $i = 1, \ldots, m$, being their dimensions. Hence, $dim(a_{ij}) = {}^{re_i}/_{x_j}$. $\mathbf{C}^\mathsf{T} = (c_{kj})_{K,n}$; $j = 1, \ldots, n$ and $k = 1, \ldots, K$ are elements of the K linear objective functions, with o_k being their dimensions. Hence, $dim(c_{kj}) = {}^{o_k}/_{x_j}$. As consideration of each stakeholder has to be negotiated they agree to subordinate their respective interests under a common utility U, which in turn makes $dim(y_k) = {}^{U}/_{o_k}$. This agreement causes a conditioned provision of resources for each of them. They offer resources relative to the importance of their objectives. Here matrix $\mathbf{B} = (b_{ik})_{m,K}$ comes into play, with $dim(b_{ik}) = \frac{re_i}{U/o_k}$. When defining $\mathbf{u}^\mathsf{T} = (u_1, \ldots, u_i, \ldots, u_m)$ with $dim(u_i) = {}^{U}/_{re_i}$, we obtain $\mathbf{C}^\mathsf{T}\mathbf{x} = (c_{kj})\mathbf{x}$ with $dim(\mathbf{C}^\mathsf{T}\mathbf{x})_k = o_k$, $\mathbf{u}^\mathsf{T}\mathbf{B} = \mathbf{u}^\mathsf{T}(b_{ik})$ with $dim(\mathbf{u}^\mathsf{T}\mathbf{B})_k = o_k$. Ultimately, $dim(\mathbf{A}\mathbf{x})_i = dim(\mathbf{B}\mathbf{y})_i = re_i$; $dim(\mathbf{u}^\mathsf{T}\mathbf{A})_j = dim(\mathbf{y}^\mathsf{T}\mathbf{C}^\mathsf{T})_j = {}^{U}/_{x_j}$.

Consequently, all dimensions in Eqs. (4) and (5′) are appropriate for respective calculations and relations. But are the decision makers alias stakeholders able to calibrate the data $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}^\mathsf{T}$ in such a way that a satisfying aggregation of their objectives $\mathbf{C}^\mathsf{T}\mathbf{x} = (\mathbf{u}^\mathsf{T}\mathbf{B})^\mathsf{T}$ is possible? Will the joint venture be successful?

5 A joint venture production of domestic appliances

Three entrepreneurs experienced in the production of domestic appliances want to join their activities to take advantage of scale effects. The first one up to now produced washing machines, the second one tumble dryers, and the third one dishwashers. As experts they know that production processes for these products are similar; and common use of a fabric, equipment, and man-power hopefully will benefit them all. To support the decision of planning common resources, they want to make use of LVOP as they have different objective functions.

To keep this application transparent, we consider $\mathbf{x}^\mathsf{T} = (x_1, x_2, x_3)$ annual production of washing machines, tumble dryers and dishwashers. All products run through the processes sheet metal forming, electric installation, assembly work, and finally packing and stocking. They estimate the activity matrix $(a_{ij})_{4,3} = \begin{pmatrix} .2 & .2 & .3 \\ .1 & .3 & .15 \\ .3 & .2 & .2 \\ .2 & .3 & .1 \end{pmatrix}$ working hours for product j in process i. The profits per unit of $\mathbf{x}^\mathsf{T} = (x_1, x_2, x_3)$ are 100 €, 80 €, 90 €, respectively. In a first naïve agreement they want to share profits making their common vector-valued objective function $\mathbf{C}^\mathsf{T} = (c_{kj})_{3,3} = \begin{pmatrix} 100/3 & 80/3 & 90/3 \\ 100/3 & 80/3 & 90/3 \\ 100/3 & 80/3 & 90/3 \end{pmatrix}$, giving an equal part of profit for each unit of the products to them all. Following the theoretical discussion in the present paper, they opt for a conditioned provision of annual resources $\mathbf{B} = (b_{ik})_{4,3} = \begin{pmatrix} 900 & 1125 & 1000 \\ 900 & 1125 & 1000 \\ 900 & 1125 & 1000 \\ 900 & 1125 & 1000 \end{pmatrix}$ of stakeholder k in working process i, $\frac{re_i}{U/o_k}$. Each of them apportions equal resources across the production processes. Entrepreneur $k = 1$ provides less resources as he/she feels aggrieved by the profit distribution. The reader might interpret the second and the third column of $\mathbf{B}$ in the same way.

All this now reads

$$\text{'Max'} \quad \mathbf{C}^\mathsf{T}\mathbf{x} = \begin{pmatrix} 100/3 & 80/3 & 90/3 \\ 100/3 & 80/3 & 90/3 \\ 100/3 & 80/3 & 90/3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

$$\text{s.t.} \quad \mathbf{A}\mathbf{x} = \begin{pmatrix} .2 & .2 & .3 \\ .1 & .3 & .15 \\ .3 & .2 & .2 \\ .2 & .3 & .1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \leqq \mathbf{B}\mathbf{y} = \begin{pmatrix} 900 & 1125 & 1000 \\ 900 & 1125 & 1000 \\ 900 & 1125 & 1000 \\ 900 & 1125 & 1000 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}$$

$$\text{'Min'} \quad \mathbf{u}^\mathsf{T}\mathbf{B} = (u_1, u_2, u_3, u_4) \begin{pmatrix} 900 & 1125 & 1000 \\ 900 & 1125 & 1000 \\ 900 & 1125 & 1000 \\ 900 & 1125 & 1000 \end{pmatrix}$$

$$\text{s.t.} \quad \mathbf{u}^\mathsf{T}\mathbf{A} = (u_1, u_2, u_3, u_4) \begin{pmatrix} .2 & .2 & .3 \\ .1 & .3 & .15 \\ .3 & .2 & .2 \\ .2 & .3 & .1 \end{pmatrix} \geqq \mathbf{y}^\mathsf{T}\mathbf{C}^\mathsf{T} = (y_1, y_2, y_3) \begin{pmatrix} 100/3 & 80/3 & 90/3 \\ 100/3 & 80/3 & 90/3 \\ 100/3 & 80/3 & 90/3 \end{pmatrix}$$

$\mathbf{x}^\mathsf{T} = (x_1, x_2, x_3) \geqq \mathbf{0}$; $\mathbf{u}^\mathsf{T} = (u_1, u_2, u_3, u_4) \geqq \mathbf{0}$; $\mathbf{y}^\mathsf{T} = (y_1, y_2, y_3) > \mathbf{0}; y_1 + y_2 + y_3 = 1$.
We observe that for all $\mathbf{x}$ and all $\mathbf{u}$, $\mathbf{C}^\mathsf{T}\mathbf{x} = (\mathbf{u}^\mathsf{T}\mathbf{B})^\mathsf{T}$ never holds. There is no satisfying aggregation of primal and dual objectives. Due to a naïve data setting, the joint venture fails. Even worse: From Section 3 we know that the non-existence of a solution of Eq. (4) results in a solution of Eq. (5′) with some non-positive y_k. We calculated the solution for (5′) with data from this section. The result is

$$\begin{aligned} (x_1, x_2, x_3) &\approx (321.29, 321.29, 8393.57) \\ (y_1, y_2, y_3) &\approx (0, 1.77, -0.77) \\ (u_1, u_2, u_3, u_4) &\approx (3.21, 3.21, 3.21, 3.21) \end{aligned}$$

indicating an embarrassing affront of stakeholder no 3 against the others. He/she does not provide resources but plans a withdrawal. Furthermore, the imperative 'Max' $\mathbf{C}^\mathsf{T}\mathbf{x}$ is in vain; the 3rd objective function is to be minimized instead of maximized, against mathematical and economical rationale – as emphasized in Section 3.2. That harms their common activities.

The sobering experience from a very naïve attitude makes the three stakeholders alter some of the data

- in that they make $\mathbf{C}^\mathsf{T} = \begin{pmatrix} 100 & 0 & 0 \\ 0 & 80 & 0 \\ 0 & 0 & 90 \end{pmatrix}$ and

- in that they calibrate better their conditioned provisions of resources $\mathbf{B} = (b_{ik})_{4,3} = \begin{pmatrix} 2000 & 1500 & 3000 \\ 800 & 2000 & 1500 \\ 4000 & 1500 & 1500 \\ 2000 & 2000 & 1000 \end{pmatrix}$.

The entries of $\mathbf{C}^\mathsf{T}$ now reflect the economic credo that each stakeholder receives the profit from his/her product exclusively. The entries in $\mathbf{B}$ now show a better adaption of resources' consumption re_i in respective working processes. With these data, the dual LVOPs read

$$\text{'Max'} \qquad \mathbf{C}^\mathsf{T}\mathbf{x} = \begin{pmatrix} 100 & 0 & 0 \\ 0 & 80 & 0 \\ 0 & 0 & 90 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

$$\text{s.t.} \qquad \mathbf{A}\mathbf{x} = \begin{pmatrix} .2 & .2 & .3 \\ .1 & .3 & .15 \\ .3 & .2 & .2 \\ .2 & .3 & .1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \leqq \mathbf{B}\mathbf{y} = \begin{pmatrix} 2000 & 1500 & 3000 \\ 800 & 2000 & 1500 \\ 4000 & 1500 & 1500 \\ 2000 & 2000 & 1000 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}$$

$$\text{'Min'} \qquad \mathbf{u}^\mathsf{T}\mathbf{B} = (u_1, u_2, u_3, u_4) \begin{pmatrix} 2000 & 1500 & 3000 \\ 800 & 2000 & 1500 \\ 4000 & 1500 & 1500 \\ 2000 & 2000 & 1000 \end{pmatrix}$$

$$\text{s.t.} \qquad \mathbf{u}^\mathsf{T}\mathbf{A} = (u_1, u_2, u_3, u_4) \begin{pmatrix} .2 & .2 & .3 \\ .1 & .3 & .15 \\ .3 & .2 & .2 \\ .2 & .3 & .1 \end{pmatrix} \geqq \mathbf{y}^\mathsf{T}\mathbf{C}^\mathsf{T} = (y_1, y_2, y_3) \begin{pmatrix} 100 & 0 & 0 \\ 0 & 80 & 0 \\ 0 & 0 & 90 \end{pmatrix}$$

$\mathbf{x}^\mathsf{T} = (x_1, x_2, x_3) \geqq \mathbf{0}$; $\mathbf{u}^\mathsf{T} = (u_1, u_2, u_3, u_4) \geqq \mathbf{0}$; $\mathbf{y}^\mathsf{T} = (y_1, y_2, y_3) > \mathbf{0}; y_1 + y_2 + y_3 = 1$.
For these two problems, we solved Eq. (4); the results are

$$\begin{aligned} (x_1, x_2, x_3) &\approx (1246.75, 3896.10, 2597.40) \\ (y_1, y_2, y_3) &\approx (0.1558, 0.5844, 0.2597) \\ (u_1, u_2, u_3, u_4) &\approx (0, 15.5844, 0, 0) \end{aligned}$$

So $\mathbf{C}^\mathsf{T}\mathbf{x} = \begin{pmatrix} 124,675 \\ 311,688 \\ 233,766 \end{pmatrix}$ is the expected profit ${}^{o_k}/_{x_j} \cdot x_j = o_k$ for the three stakeholders. What their respective investments is concerned we have $(\mathbf{u}^\mathsf{T}\mathbf{B})^\mathsf{T} = \begin{pmatrix} 124,675 \\ 311,688 \\ 233,766 \end{pmatrix}$, ${}^{U}/_{re_i} \cdot \frac{re_i}{{}^{U}/_{o_k}} = o_k$. Both values coincide, as it should be. Another interesting information are the resources which each stakeholder must put into the production processes, namely $\begin{pmatrix} 2000 \\ 800 \\ 4000 \\ 2000 \end{pmatrix} y_1 = \begin{pmatrix} 311.6 \\ 124.6 \\ 623.2 \\ 311.6 \end{pmatrix}$; $\begin{pmatrix} 1500 \\ 2000 \\ 1500 \\ 2000 \end{pmatrix} y_2 = \begin{pmatrix} 876.6 \\ 1108.8 \\ 876.6 \\ 1168.8 \end{pmatrix}$; $\begin{pmatrix} 3000 \\ 1500 \\ 1500 \\ 1000 \end{pmatrix} y_3 = \begin{pmatrix} 777.6 \\ 388.8 \\ 388.8 \\ 259.2 \end{pmatrix}$. They must check whether they are able to provide such resources in addition to their still remaining activities, producing washing machines, tumble dryers, and dishwashers.

6 Conclusions

In Wikipedia, we find the definition: "In mathematics, a duality translates concepts, theorems or mathematical structures into other concepts, theorems or structures, ..." In the context of this contribution, the mathematical structures are a linear vector maximum problem and its dual linear vector minimum problem. Duality should enrich the knowledge about such mathematical structures. That is exactly what we observe for dual LVOPs.

We describe the planning process for a joint venture of entrepreneurs which want to organize common production in a new factory. In doing so, they must harmonize their (different) objectives and their (different) investments. Here, LVOP duality theory becomes effective. And this theory demands a set of compatible data like objectives, activities, and investments. In an application, incompatible data imply in a failure of the venture; compatible data end up in a successful venture. The achievement of compatible data is a difficult task in a situation with many entrepreneurs, activities and resources.

Is there a key for (iteratively) generating an economic expedient data set; and do these data reflect the actors' readiness to cooperate in the venture? Our future work must give an answer to these questions,

References

A.Y. Azimov. Duality for set-valued multiobjective optimization problems. *Journal of Optimization Theory and Applications*, 137:75–88, 2008. doi: 10.1007/s10957-007-9314-x.

A. Charnes, W.W. Cooper, and A. Henderson. *An Introduction to Linear Programming*. John Wiley & Sons, New York, 1953.

L. Collatz and W. Wetterling. *An Introduction to Linear Programming*. Springer, Berlin, Heidelberg, 1966. doi: https://doi.org/10.1007/978-3-662-00452-4.

D. Gale, H.W. Kuhn, and A.W. Tucker. Linear programming and the theory of games. In T. C. Koopmans, editor, *Activity Analysis of Production and Allocation*, pages 317–329. John-Wiley and Sons, New York, London, 1951.

A. Gardi, R. Sabatini, and S. Ramasamy. Multi-objective optimisation of aircraft flight trajectories in the atm and avionics context. *Progress in Aerospace Sciences*, 83:1–36, 2016. doi: https://doi.org/10.1016/j.paerosci.2015.11.006.

G. Hadley. *Linear Programming*. Addison-Wesley Publishing Company, Reading, MA et al., 1962.

A.H. Hamel. A fenchel-rockafellar duality theorem for set-valued optimization. *Optimization*, 60(8-9):1023–1043, 2011. doi: 10.1080/02331934.2010.534794.

F. Heyde and A. Löhne. Solution concepts in vector optimization: a fresh look at an old story. *Optimization*, 60(12):1421–1440, 2011. doi: 10.1080/02331931003665108.

F. Heyde and C. Schrage. Continuity concepts for set-valued functions and a fundamental duality formula for set-valued optimization. *Journal of Mathematical Analysis and Applications*, 397(2):772–784, 2013. doi: https://doi.org/10.1016/j.jmaa.2012.08.019.

Y. Ijiri. *Management Goals and Accounting for Control*. North Holland Publishing Company, Amsterdam, 1965.

H. Isermann. *Lineare Vektoroptimierung (German)*. Dissertation, Universität Regensburg, Germany, 1974.

A. Kleine and A. Dellnitz. Allocation of seminar applicants: A goal programming approach. *Journal of Business Economics*, 87(1):927–941, 2017.

J.S.H. Kornbluth. Duality, Indifference and Sensitivity Analysis inr Multiple Objective Linear Programming. *Journal of the Operational Research Society*, 25:599–614, 1974.

H.W. Kuhn and A.W. Tucker. Nonlinear Programming. In *Proceedings of the Second Berkeley Symposium on Mathematical Statistics and Probability*, pages 481–492. University of California Press, 1951.

O.L. Mangasarian. *Nonlinear Programming*. North Holland Publishing Company, McGraw-Hill, New York, 1969.

N. Mansour, M.S. Cherif, and W. Abdelfattah. Multi-objective imprecise programming for financial portfolio selection with fuzzy returns. *Expert Systems with Applications*, 138: 112810, 2019. doi: https://doi.org/10.1016/j.eswa.2019.07.027.

T.S. Motzkin. *Beiträge zur Theorie der linearen Ungleichungen*. Universität Basel, Basel, 1936.

W. Rödder. Satisfying Aggregation of Objectives by Duality. In G. Fandel and T. Gal, editors, *Lecture Notes in Economics and Mathematical Systems 177*, pages 317–329. Springer, Berlin, Heidelberg, New York, 1980.

W. Rödder and H. Leberling. Some notes on duality for linear vector optimization problems. *Working paper no 76/11, Institut für Wirtschaftswissenschaften, RWTH Aachen*, 1976.

K.P. Schönfeld. *Effizienz und Dualität in der Aktivitätsanlyse (German)*. Dissertation, Freie Universität Berlin, Germany, 1964.

J. von Neumann and O. Morgenstern. *Theory of Games and Economic Behavior*. Princeton University Press, Princeton, 1944.

P.J. Williams and W.L. Kendall. A guide to multi-objective optimization for ecological problems with an application to cackling goose management. *Ecological Modelling*, 343:54–67, 2017. doi: https://doi.org/10.1016/j.ecolmodel.2016.10.010.

M.M. Yenisey and B. Yagmahan. Multi-objective permutation flow shop scheduling problem: Literature review, classification and current trends. *Omega*, 45:119–135, 2014. doi: https://doi.org/10.1016/j.omega.2013.07.004.

S. Zajac and S. Huber. Objectives and methods in multi-objective routing problems: a survey and classification scheme. *European Journal of Operational Research*, 290(1): 1–25, 2021. doi: https://doi.org/10.1016/j.ejor.2020.07.005.

Appendix A1

We show a set of LVOPs like in (3a) and (3b) with the characteristics mentioned at the end of Section 3.2.

i)

$$\begin{aligned} \text{'Max'} \quad & \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \\ \text{s.t.} \quad & \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \leqq \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \end{aligned}$$

$$\begin{aligned} \text{'Min'} \quad & (u_1, u_2) \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \\ \text{s.t.} \quad & (u_1, u_2) \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \geqq (y_1, y_2) \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \end{aligned}$$

$\mathbf{x}^\mathsf{T} = (x_1, x_2) \geqq \mathbf{0}$; $\mathbf{u}^\mathsf{T} = (u_1, u_2) \geqq \mathbf{0}$; $\mathbf{y}^\mathsf{T} = (y_1, y_2) > \mathbf{0}$; $y_1 + y_2 = 1$.

Both LVOPs are dual but infeasible for all $\mathbf{y}$.

ii)

$$\begin{aligned} \text{'Max'} \quad & \begin{pmatrix} 1 \\ 1 \end{pmatrix} x \\ \text{s.t.} \quad & 1 \cdot x \leqq (1,1) \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \end{aligned}$$

$$\begin{aligned} \text{'Min'} \quad & u(1,1) \\ \text{s.t.} \quad & u \cdot 1 \geqq (y_1, y_2) \begin{pmatrix} 1 \\ 1 \end{pmatrix} \end{aligned}$$

$x \geqq 0$; $u \geqq 0$; $\mathbf{y}^\top = (y_1, y_2) > \mathbf{0}; y_1 + y_2 = 1$.

Both LVOPs are dual and feasible. For all $\mathbf{y}$, equality holds: $\begin{pmatrix} 1 \\ 1 \end{pmatrix} x = \big(u(1,1)\big)^\top$ for $x = 1$, $u = 1$.

iii)

$$\begin{aligned} \text{'Max'} \quad & \begin{pmatrix} 1 \\ 2 \end{pmatrix} x \\ \text{s.t.} \quad & 1 \cdot x \leqq (1,1) \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \end{aligned}$$

$$\begin{aligned} \text{'Min'} \quad & u(1,1) \\ \text{s.t.} \quad & u \cdot 1 \geqq (y_1, y_2) \begin{pmatrix} 1 \\ 2 \end{pmatrix} \end{aligned}$$

$x \geqq 0$; $u \geqq 0$; $\mathbf{y}^\top = (y_1, y_2) > \mathbf{0}; y_1 + y_2 = 1$.

Both LVOPs are dual for all $\mathbf{y}$ but for f.e. $x = 1$ and $u = (y_1, y_2) \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ equality of primal and dual function values fails. Following the reasoning in Section 3.2, this implies in Eq. (5′):

$\exists (x, u, \mathbf{y}) : u\mathbf{A} \geqq (y_1, y_2)\mathbf{C}^\top$, $\mathbf{A}x \leqq \mathbf{B} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$, $\mathbf{C}^\top x \geq (u\mathbf{B})^\top$; choose $x = 1$, $u = {}^3\!/_4$, $y_1 = {}^5\!/_4$, $y_2 = -{}^1\!/_4$.

iv)

$$\begin{aligned}
&\text{'Max'} && \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \\
&\text{s.t.} && \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \leqq \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}
\end{aligned}$$

$$\begin{aligned}
&\text{'Min'} && (u_1, u_2) \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \\
&\text{s.t.} && (u_1, u_2) \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \geqq (y_1, y_2) \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}
\end{aligned}$$

$\mathbf{x}^\mathsf{T} = (x_1, x_2) \geqq \mathbf{0}$; $\mathbf{u}^\mathsf{T} = (u_1, u_2) \geqq \mathbf{0}$; $\mathbf{y}^\mathsf{T} = (y_1, y_2) > \mathbf{0}; y_1 + y_2 = 1.$

For $y_1 \cong 1$, we get for f.e. $\mathbf{x} \ : \ x_1 + x_2 = 1$ and hence all convex combinations of $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$ and $\begin{pmatrix} 2 \\ 1 \end{pmatrix}$ for efficient objective function values.

For $y_1 \cong 1$, we get for f.e. $\mathbf{u} \ : \ u_1 + u_2 \cong 2$. Efficient value of $(u_1, u_2) \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$ then is $\begin{pmatrix} \approx 2 \\ \approx 2 \end{pmatrix}$.

The choice of $y_1 = y_2 = 1/2$, however, does not change the situation for the primal but makes efficient value of the dual objective value equal $(3/2, 3/2)$.

Resuming: There is no equality in the case $y_1 \cong 1$, but equality for $y_1 = y_2 = 1/2$. Reachability of equality depends on parameter $\mathbf{y}$. The following figures might make this result more transparent.

Figure 2: Primal and dual f.e. objective function values, $y_1 \cong 1$

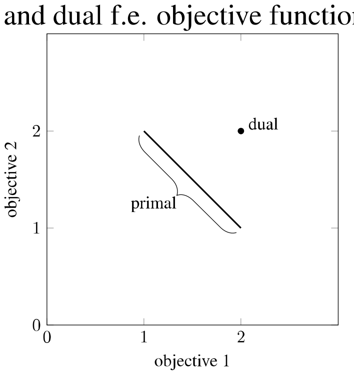
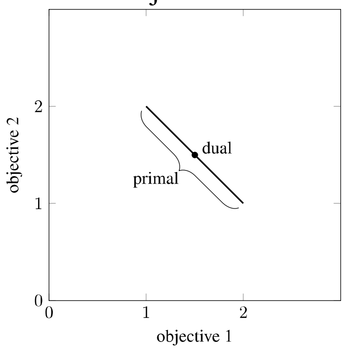


Figure 3: Primal and dual f.e. objective function values, $y_1 = y_2 = 1/2$



Bisher erschienene Research Papers:

Nr. 1 | Februar 2017
Berlemann, Michael; Christmann, Robin.
The Role of Precedents on Court Delay. Evidence from a Civil Law Country.

Nr. 2 | September 2017
Matthes, Roland.
A note on the Saito-Kurokawa lift for Hermitian forms.

Nr. 3 | Februar 2018
Christmann, Robin.
Prosecution and Conviction under Hindsight Bias in Adversary Legal Systems

Nr. 4 | März 2019
Broere, Mark; Christmann, Robin.
Takeovers, Shareholder Litigation, and the Free-riding Problem

Nr. 5 | März 2019
Matthes, Roland.
Vector-valued Cusp Forms and Orthogonal Modular Forms

Nr. 6 | März 2020
Matthes, Roland.
A Geometric View on Linear Regression and Correlation Tests

Nr. 7 | März 2020
Matthes, Roland.
A Note on the Geometry of Partial Correlation and the Grassmannian

Nr. 8 | April 2020
Christmann, Robin; Kirstein, Roland.
You Go First! Coordination Problems and the Standard of Proof in Inquisitorial Prosecution

Nr. 9 | Juli 2021
Christmann, Robin.
Plea Bargaining and Investigation Effort: Inquisitorial Criminal Procedure as a Three Player Game

Nr. 10 | Dezember 2023
Afsharian, Mohsen.
Data Science Essentials for Business: Exploring Analytics and Data Scientists' Contributions

Nr. 11 | Dezember 2023
Broere, Mark; Matthes, Roland.
The Error of Present Value Discounting Conventions in the Case of Uniform Intra-Period Cash Flow

Nr. 12 | Dezember 2023
Rödder, Wilhelm; Dellnitz, Andreas.
Planning a Joint Venture: Decision Support by Linear Vector Duality Theory

Nr. 13 | Dezember 2023
Christmann, Robin; Klein, Dennis.
Compliance, Liability and Corporate Criminal Law: A Setup for a Three-Player Inspection Game

Leibniz FH

Leibniz-Fachhochschule
Expo Plaza 11
30539 Hannover

leibniz-fh.de